Deformations of Coisotropic Submanifolds of Jacobi Manifolds

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Coisotropic submanifolds play a distinguished role in Poisson geometry:

- Lagrangian submanifolds of symplectic manifolds,
- first class constraints in Hamiltonian mechanics,
- reduction of Poisson manifolds,
- morphisms of Poisson manifolds,
- branes in Poisson $\sigma$-models.

**Remark**

The *moduli spaces* of coisotropic submanifolds under Hamiltonian and Poisson diffeomorphisms is of a special interest! The *deformation problem* for coisotropic submanifolds has been first studied in

- [Oh, Park 2005] in the symplectic setting,
- [Schätz 2009] in the Poisson setting,
- [Lê, Oh 2012] in the lcs setting.
Jacobi manifolds were introduced in [Kirillov 1976] and, independently, [Lichnerowicz 1978]. They encompass

- Poisson manifolds,
- lcs manifolds,
- contact manifolds.

One can define coisotropic submanifolds of Jacobi manifolds and they play a special role in Jacobi geometry:

- Legendrian submanifolds of contact manifolds,
- reduction of Jacobi manifolds,
- morphisms of Jacobi manifolds.

**Aim of the Talk**

I will discuss the moduli problem of coisotropic submanifolds of a Jacobi manifold under Hamiltonian diffeomorphisms. In particular, I will describe an $L_\infty$-algebra governing the deformation problem.
Definition

A Jacobi bundle over a manifold $M$ is a line bundle $L \to M$ equipped with a Jacobi bracket, i.e. a first order, skew-symmetric bi-differential operator $\{-,-\} : \Gamma(L) \times \Gamma(L) \to \Gamma(L)$ satisfying the Jacobi identity. A Jacobi manifold is a manifold together with a Jacobi bundle over it.

Remark

For every $\lambda \in \Gamma(L)$ there is an associated Hamiltonian vector field $X_\lambda$ implicitly defined by

$$\{\lambda, f\mu\} = f\{\lambda, \mu\} + X_\lambda(f)\mu, \quad \lambda, \mu \in \Gamma(L), \quad f \in C^\infty(M).$$

Proposition

1. Hamiltonian vector fields generate an integrable distribution $K$.
2. Every leaf of $K$ is equipped with either a contact or a lcs structure.
3. The contact/lcs foliation knows everything about $(M, L, \{-,-\})$. 
Let \((M, L, \{-, -\})\) be a Jacobi manifold.

**Definition**

A submanifold \(S \subset M\) is **coisotropic** if \(X_\lambda\) is tangent to \(S\) for all \(\lambda \in \Gamma(L)\) vanishing on \(S\). Equivalently, \(S\) is coisotropic if \(\{\lambda, \mu\}\) vanishes on \(S\) for all \(\lambda, \mu \in \Gamma(L)\) vanishing on \(S\).

Examples are:
- coisotropic submanifolds of Poisson manifolds,
- coisotropic submanifolds of contact manifolds,
- coisotropic submanifolds of lcs manifolds,
- graphs of Jacobi maps.

**Proposition**

A submanifold \(S \subset M\) is coisotropic iff its intersection with every characteristic leaf \(\mathcal{K}\) is coisotropic in \(\mathcal{K}\).
Let \((M, L, \{-, -\})\) be a Jacobi manifold.

**Remark**

There is a unique Lie algebroid structure on \(J^1L \to M\) such that

\[
[j^1\lambda, j^1\mu] = j^1\{\lambda, \mu\},
\]

\[
\rho(j^1\lambda) = X_\lambda.
\]

Additionally, there is a unique action of \(J^1L\) on \(L\) such that

\[
j^1\lambda \cdot \mu = \{\lambda, \mu\}.
\]

**Proposition**

\(S \subset M\) is coisotropic iff \(N^*S \otimes L|_S \to T^*M \otimes L \to J^1L\) is the inclusion of a Lie subalgebroid. In this case there is a cochain complex

\[
0 \to \Gamma(L|_S) \xrightarrow{\partial_S} \Gamma(NS) \xrightarrow{\partial_S} \Gamma(\wedge^2(NS \otimes L^*_|S) \otimes L|_S) \xrightarrow{\partial_S} \cdots.
\]
I want to describe the space of coisotropic submanifolds \textit{locally around a given point }S. 

\[
\{\text{coisotropic submanifolds } C^1\text{-close to } S\} \parallel \{\text{coisotropic sections of a tubular neighborhood } \pi : E \rightarrow S \text{ of } S\}
\]

**Local Setting**

- A vector bundle \( \pi : E \rightarrow S \),
- A line bundle \( L_S \rightarrow S \) and the pull-back bundle \( L := \pi^*L_S \rightarrow E \),
- A Jacobi bracket on \( \Gamma(L) \) such that \( 0 : S \hookrightarrow E \) is coisotropic.

**Definition**

A coisotropic deformation of \( S \) is a section \( s : S \hookrightarrow E \) of \( \pi \) such that \( s(S) \) is a coisotropic submanifold.
Hamiltonian diffeomorphisms are those generated by Hamiltonian vector fields $X_\lambda$, with $\lambda \in \Gamma(L)$. They should be understood as inner automorphisms of the Jacobi manifold, and act on coisotropic submanifolds.

**Definition**

Two coisotropic deformations are *Hamiltonian equivalent* if they are intertwined by an Hamiltonian diffeomorphism ($C^1$-close to id).

**Theorem**

1. The deformation problem of a coisotropic submanifold $S$ in a Jacobi manifold is controlled by a certain $L_\infty$-algebra $\mathfrak{g}(S)$.
2. $\mathfrak{g}(S)$ is uniquely defined by $S$ up to $L_\infty$-isomorphisms.

**Warning!**

In fact, for generic Jacobi manifolds, I can only treat formal coisotropic deformations. Otherwise I have to impose an *entireness condition* on the Jacobi bracket.
L∞-algebras are Lie algebras up to homotopy and generalize DGLAs. Let V be a graded vector space.

**Definition**

An L∞-algebra structure on V is a sequence of graded maps:

\[ l_k : V^{\wedge k} \longrightarrow V[2 - k], \]

satisfying a sequence of coherence conditions:

\[
\sum_{i+j=k} (-)^{ij} \sum_{\sigma \in S_{i,j}} \epsilon(\sigma, x) l_{j+1}(l_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(i+j)}).
\]

1. \( l_2^2(x) = 0, \)
2. \( l_1 l_2(x, y) = l_2(l_1 x, y) \pm l_2(x, l_1 y), \)
3. \( l_2(x, l_2(y, z)) \pm l_2(y, l_2(z, x)) \pm l_2(y, l_2(z, x)) = l_1 l_3(x, y, z) + l_3(l_1 x, y, z) \pm l_3(x, l_1 y, z) \pm l_3(x, y, l_1 z), \)
4. \( \ldots \)
Deformation Theory via $L_\infty$-algebras

Let $(g, \{l_k\})$ be an $L_\infty$-algebra.

**Definition**

1. A **Maurer-Cartan element** (MC) is $x \in g^1$ such that
   $$\sum_{k=1}^{\infty} \frac{1}{k!} l_k(x, \ldots, x) = 0.$$

2. Two MC elements $x_0, x_1$ are **gauge equivalent** if they are interpolated by a family $\{x_t\}$ and there is a family $\{y_t\} \subset g^0$ such that
   $$\frac{dx_t}{dt} = \sum_{k=0}^{\infty} \frac{1}{k!} l_{k+1}(x_t, \ldots, x_t, y_t).$$

**Definition**

An $L_\infty$-algebra $(g, \{l_k\})$ **controls** a deformation problem if

1. deformations are in 1-1 correspondence with MC elements of $g$,
2. two deformations are equivalent iff the corresponding MC elements are gauge equivalent.
Let $S$ be a coisotropic submanifold of a Jacobi manifold $(M, L, \{-,-\})$. Choose a tubular neighborhood $\pi : E \to S$ and adopt the local setting.

**Remark**

First order multidifferential operators $\Gamma(L) \times \cdots \times \Gamma(L) \to \Gamma(L)$ form a graded Lie algebra $\text{Der}^\cdot L$ with a Schouten-like bracket $[[\ - , - ]]$. The Jacobi bracket can be seen as a MC element in $\text{Der}^\cdot L$. Denote it by $\mathcal{J}$.

**Proposition**

1. There is a canonical projection $P : \text{Der}^\cdot L \to \Gamma(\wedge^\cdot (E \otimes L_S^*) \otimes L_S)$ with right inverse $I$ depending on $\pi$.
2. Voronov’s formula [Voronov 2005]

$$l_k(x_1, \ldots, x_k) := \pm P[[\cdots [\mathcal{J}, I(x_1)], \cdots], I(x_k)]$$

defines an $L_\infty$-algebra $(g(S), \{l_k\})$ with $g(S) = \Gamma(\wedge^\cdot (E \otimes L_S^*) \otimes L_S)$.
3. $(g(S), \{l_k\})$ is independent of the choice of a tubular neighborhood up to $L_\infty$-isomorphisms.
Let $S$ be a coisotropic submanifold of a Jacobi manifold $(M, L, \{-, -\})$.

**Theorem**

The $L_\infty$-algebra $(\mathfrak{g}(S), \{l_k\})$ controls the deformation problem of $S$, i.e.

1. coisotropic deformations of $S$ are in 1-1 correspondence with MC elements of $(\mathfrak{g}(S), \{l_k\})$, and
2. two deformations are Hamiltonian equivalent iff the corresponding MC elements are gauge equivalent.

**Corollary**

1. Non-trivial infinitesimal deformations are in 1-1 correspondence with $H^1(\Gamma(\cdot (\mathcal{N}S \otimes L_S^*) \otimes L_s), \partial S)$,
2. obstruction to the prolongation of an infinitesimal deformation to a formal one live in $H^2(\Gamma(\cdot (\mathcal{N}S \otimes L_S^*) \otimes L_s), \partial S)$. 
The Contact Case

Let \((M, C)\) be a non-necessarily coorientable contact manifold and let \(S \subset M\) be a coisotropic submanifold.

**Remark**

I assume that \(TS \cap C\) has constant rank.

1. \(L = TM/C\) is a Jacobi bundle and \(S\) is coisotropic wrt it,
2. either \(S\) is Legendrian, or \(C_S := TS \cap C\) is a precontact distribution,
3. the characteristic distribution of \((S, C_S)\) has constant rank.

**Coisotropic Neighborhood Theorem**

A neighborhood of \(S\) does only depend on the intrinsic precontact geometry of \(S\) up to contactomorphisms.

**Corollary**

The \(L_\infty\)-algebra of \(S\) does only depend on the intrinsic precontact geometry of \(S\) up to \(L_\infty\)-isomorphisms.
The $L_\infty$-algebra of $S$ can be made more explicit in the contact case.

**Proposition**

Assume $S$ is not Legendrian. Let $\mathcal{F}$ be the characteristic foliation of $(S, C_S)$ and let $R$ be the curvature of a complementary distribution to $T\mathcal{F}$.

$$g(S) = \Omega(\mathcal{F}, L|_S), \quad \text{and} \quad t_k = \begin{cases} \frac{d\mathcal{F}}{\mathcal{O}(R^{k-2})} & \text{for } k = 1 \\ \mathcal{O}(R^{k-2}) & \text{for } k > 1 \end{cases}.$$ 

**Corollary**

1. Non-trivial infinitesimal deformations are in $H^1(\mathcal{F}, L|_S)$,
2. obstruction to the prolongation live in $H^2(\mathcal{F}, L|_S)$.

**Corollary**

The $L_\infty$-algebra of the flowout of a Legendrian submanifold under an Hamiltonian vector field is a DGLA.
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Thank you!