From Finite to Infinite: which strategy for groups of high cardinality?

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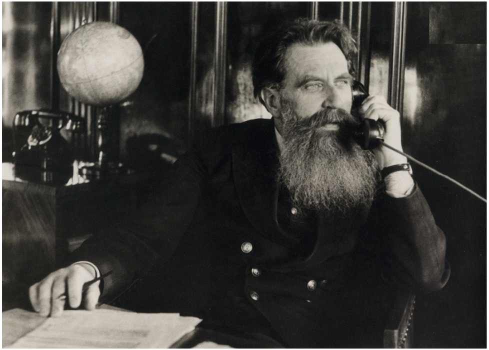


Otto Yu. Šmidt (1891–1956), wrote the first monograph on group theory which was not limited to the finite case.
He was also a famous explorer and a Heroe of Soviet Union

Finiteness conditions in group theory - The origins

(**I. Schur**, 1902) *Let G be a group whose center $Z(G)$ has finite index. Then the commutator subgroup G' of G is finite.*

Issai Schur (1875–1941)

Finiteness conditions in group theory - The origins

(**S.N. Černikov**, 1940) *Let G be a soluble group satisfying the minimal condition on subgroups. Then G contains an abelian subgroup of finite index.*

Sergei N. Černikov (1912–1987)

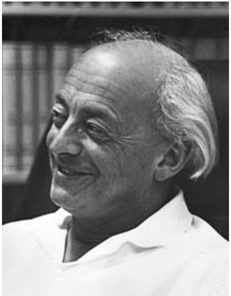
Finiteness conditions in group theory - The origins

(**P. Hall**, 1954) *Let G be a soluble group satisfying the maximal condition on normal subgroups. Then G is finitely generated.*

Philip Hall (1904–1982)

Finiteness conditions in group theory - The origins

(**R. Baer**, 1968) *Let G be a soluble group whose abelian subgroups are minimax. Then G is minimax.*



Reinhold Baer (1902–1979)

Finiteness conditions in group theory - The origins

The same result was independently proved in the same year by D.I. Zaicev

Dmitry I. Zaicev (1942–1990)

Finiteness conditions in group theory - The origins

(**A.I. Mal'cev**, 1951) *Let G be a locally nilpotent group whose abelian subgroups have finite rank. Then G has finite rank (and it is hypercentral).*

Anatoly I. Mal'cev (1909–1967)

Finiteness conditions in group theory - The origins

(**V.S. Čarin**, 1957) *Let G be a torsion-free locally soluble group of finite rank. Then G is soluble.*

Viktor S. Čarin (1919–2008)

Finiteness conditions in group theory
a relevant development

(**B.H. Neumann** 1955)
(a) *A group G has finite conjugacy classes of subgroups if and only if its centre $Z(G)$ has finite index.*
(b) *Every subgroup of a group G has finite index in its normal closure if and only if the commutator subgroup G' is finite.*

Bernhard H. Neumann (1909–2002)

On the other hand, finiteness conditions cannot have a crucial role in the study of infinite groups,
because most of the groups which are in some sense large cannot be analyzed using this tool

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For instance, all soluble groups of finite rank are countable

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Of course, it must be specified what is meant by “large” and “small” in group theory

Let $\mathfrak{X}$ be a class of groups

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- If N is a normal subgroup of an $\mathfrak{X}$-group G, then at least one of the groups N and G/N belongs to $\mathfrak{X}$
- No finite cyclic group lies in $\mathfrak{X}$

Natural examples of classes of large groups

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- The class of infinite groups

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- The class of infinite groups
- The class of groups of infinite rank

Natural examples of classes of large groups

- The class of infinite groups
- The class of groups of infinite rank
- The class of uncountable groups

Let θ be a property pertaining to subgroups of a group

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Let θ be a property pertaining to subgroups of a group

θ is called absolute if in any group G
all subgroups isomorphic to some θ-subgroup of G
are likewise θ-subgroups

θ is called an embedding property if in any group G all images of θ-subgroups under automorphisms of G likewise have the property θ

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Normality is an embedding property which is not absolute

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We shall say that a group class $\mathfrak{X}$ controls θ
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If G is any group containing some $\mathfrak{X}$-subgroup
and all $\mathfrak{X}$-subgroups of G have the property θ,
then θ holds for all subgroups of G

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This definition can also be given inside
a fixed universe $\mathfrak{U}$

The class of cyclic groups controls periodicity

The class of cyclic groups controls periodicity

The class of finitely generated groups controls commutativity

The class of cyclic groups controls periodicity

The class of finitely generated groups controls commutativity

On the other hand, neither nilpotency nor solubility
are controlled by the class of finitely generated groups

Normality is controlled by the class of finitely generated groups
and even by that of cyclic groups

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But most of the relevant embedding properties (subnormality, for instance) cannot be controlled by the class of finitely generated groups

This phenomenon essentially is a consequence of the fact that finitely generated groups are too *small*

Let $\mathfrak{X}$ be a class of large groups,
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Since every group containing an $\mathfrak{X}$-subgroup lies in $\mathfrak{X}$,
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Since every group containing an $\mathfrak{X}$-subgroup lies in $\mathfrak{X}$,
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The first natural example of a class of large groups is the class $\mathfrak{I}$ consisting of all infinite groups

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The consideration of the locally dihedral 2-group shows that normality cannot be controlled by the class $\mathfrak{I}$, even within the universe of periodic metabelian groups

How large should be $\mathfrak{X}$-groups in order to obtain that the class $\mathfrak{X}$ controls embedding properties?

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Problem

Find natural classes of large groups which control all relevant embedding properties, at least inside suitable universes

A group G ha finite rank r
if every finitely generated subgroup of G
can be generated by at most r elements,
and r is the smallest positive integer with such property

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Groups of infinite rank form a class of large groups

(M.J. Evans and Y. Kim, 2004)

The class of groups of infinite rank controls normality
in the universe $\mathfrak{S}$ of soluble groups

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Thus if all subgroups of infinite rank
of a soluble group G are normal,
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The class of groups of infinite rank controls normality in the universe $\mathfrak{S}$ of soluble groups

Thus if all subgroups of infinite rank of a soluble group G are normal, then either G has finite rank or all its subgroups are normal

In the same paper Evans and Kim proved also that the class of groups of infinite rank controls subnormality with defect at most k in the universe $\mathfrak{S}$

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at least in the universe of soluble groups

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This project is now positively completed

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Problem

Find other classes of large groups which control
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The class of uncountable groups is of this type?

The study of the control capability
of groups of high cardinality
is the research project of my team for next years

Shelah has proved that there exist groups of cardinality $\aleph_1$ in which all proper subgroups are countable

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Groups of this type (Jónsson groups) play a role corresponding to that of Tarski groups in the countable case

On the other hand, Jónsson groups are perfect and simple over the centre, so that they can be avoided working within the universe of locally soluble groups

Observe that, while conditions related to the rank are of algebraic nature, those related to cardinality are purely set-theoretic

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On the other hand, the basic considerations on finite groups
depend on arithmetic arguments;
Lagrange's theorem, Sylow's theorem, the results of Burnside
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Lagrange's theorem, Sylow's theorem, the results of Burnside and the celebrated theorem of Feit-Thompson

Therefore in some sense the point of view just described provides a kind of unity between finite and infinite groups

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Then every uncountable subgroup of U is not abelian, and so is normal

It follows that the class of uncountable groups cannot control normality, even in the universe of nilpotent groups

The control of normality in uncountable groups

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M. De Falco - F. de Giovanni - H. Heineken - C. Musella
"Normality in uncountable groups"
(in preparation)

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Let $\aleph$ be an uncountable regular cardinal, and let G be a group of cardinality $\aleph$ in which all subgroups of cardinality $\aleph$ is normal.
Then all subgroups of G are normal,
provided that one of the following conditions holds:

- G contains an abelian subgroup of cardinality $\aleph$
- G is residually of cardinality strictly smaller than $\aleph$

The structure of groups whose uncountable subgroups are normal is in general quite restricted

The structure of groups whose uncountable subgroups are normal is in general quite restricted

Let $\aleph$ be an uncountable regular cardinal, and let G be a locally soluble group of cardinality $\aleph$ in which all subgroups of cardinality $\aleph$ are normal. Then G is a 2-Engel group. In particulare G is nilpotent of class at most 3, and its commutator subgroup G' is abelian of cardinality strictly smaller than $\aleph$.

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In this situation a sharper description of G' can be given:
G' is the direct product of a p-group and a group
of order at most 2, if G is periodic,
and G' is divisible if G is torsion-free

The control of absolute property in uncountable groups

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F. de Giovanni - M. Trombetti
"Nilpotency in uncountable groups"
(submitted to *J. Austral. Math. Soc.*)

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Let $\aleph$ be an uncountable regular cardinal, and let G be a group of cardinality $\aleph$ with no infinite simple homomorphic images. If all proper subgroups of G of cardinality $\aleph$ is nilpotent, then G itself is nilpotent.

The control of absolute properties in uncountable groups

The control of absolute properties in uncountable groups

F. de Giovanni - M. Trombetti

"Uncountable groups with restrictions on subgroups of large cardinality"

J. Algebra (2016)

The control of absolute properties in uncountable groups

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"Uncountable groups with restrictions on subgroups of large cardinality"
J. Algebra (2016)

Let $\aleph$ be an uncountable regular cardinal, and let G be a group of cardinality $\aleph$ with no infinite simple homomorphic images. If all proper subgroups of G of cardinality $\aleph$ have finite conjugacy classes, Then G itself has finite conjugacy classes

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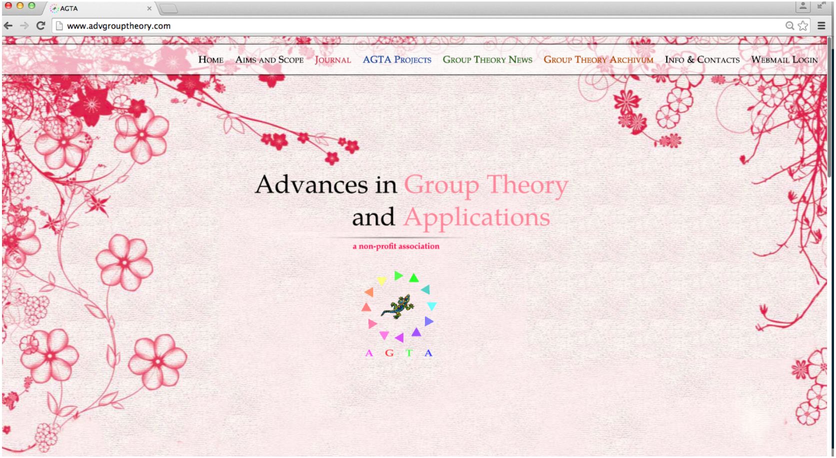
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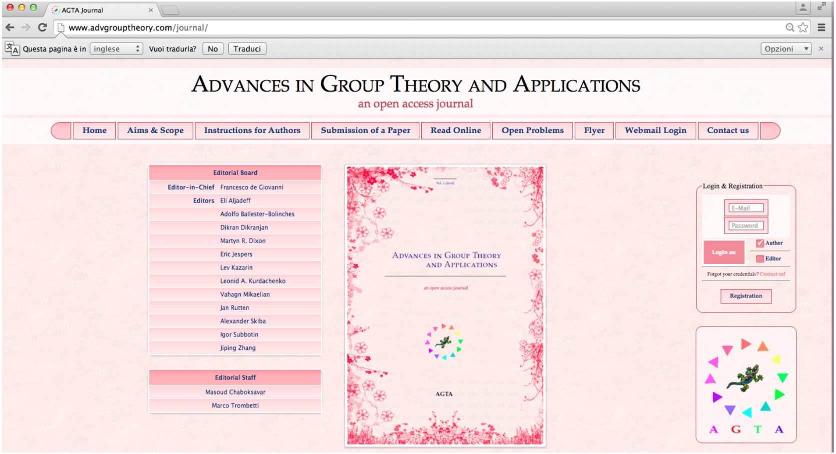
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