Representation Growth of Arithmetic Groups

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If the sequence

$$R_N(G) = \sum_{n=1}^{N} r_n(G) \text{ for } N \in \mathbb{N},$$

is bounded by a polynomial in N, the group G is said to have *polynomial representation growth* (PRG).

Representation zeta function

The representation growth of a rigid group can be studied by means of the *representation zeta function*, namely, the Dirichlet series

$$\zeta_G(s) = \sum_{n=1}^{\infty} r_n(G) n^{-s},$$

where s is a complex variable.

Abscissa of convergence

Definition

The *abscissa of convergence* $\alpha(G)$ of the series $\zeta_G(s)$ is the infimum of all $\alpha \in \mathbb{R}$ such that $\zeta_G(s)$ converges on the complex half-plane $\{s \in \mathbb{C} \mid \Re(s) > \alpha\}$

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Proposition

Let G have PRG. The abscissa of convergence $\alpha(G)$ is the smallest value such that

$$R_N(G) = O(1 + N^{\alpha(G)+\varepsilon})$$

for every $\varepsilon \in \mathbb{R}_{>0}$

Larsen and Lubotzky conjecture

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Conjecture (Larsen and Lubotzky, 2008)

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- Using p-adic integration and approximative Clifford theory, the same authors proved Larsen and Lubotzky's conjecture for groups of type A_2.

Definition

An *arithmetic group* is a group Γ which is commensurable to $\mathrm{H}(\mathcal{O})$, where H is a connected, simply connected semisimple linear algebraic group defined over a number field k and $\mathcal{O}$ is the the ring of integers in k.

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We make the following simplification: from now on an arithmetic group is $\mathrm{H}(\mathcal{O})$ for H and $\mathcal{O}$ as above.

Congruence subgroups

Definition

Let $\Gamma = \mathsf{H}(\mathcal{O})$ be and arithmetic group with and $\mathcal{O}$ as above and $\mathsf{H} \leq \mathrm{GL}_d$ for some $d \in \mathbb{N}$. A principal congruence subgroup of level m of Γ is $\Gamma \cap I_d + \mathrm{Mat}_d(\mathfrak{p}^m)$ for $\mathfrak{p}$ a prime ideal in $\mathcal{O}$.

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Definition (Congruence subgroup)

A subgroup of and arithmetic group Γ is called a congruence subgroup when it contains a principal congruence subgroup.

CSP

Definition (Congruence subgroup property)

Let S be the set of archimedean places of $\mathcal{O}$. We say that an arithmetic group $\Gamma = \mathsf{H}(\mathcal{O})$ has the *weak congruence subgroup property* (wCSP) when the map

$$\widehat{\mathsf{H}(\mathcal{O})} \to \mathsf{H}(\widehat{\mathcal{O}})$$

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Theorem (Lubotzky and Martin, 2004)

Let Γ be an arithmetic group in characteristic 0. Then Γ has PRG if and only if it has the wCSP.

Euler products

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Let $\Gamma = \mathrm{H}(\mathcal{O})$, and let S be the set of archimedean places in $\mathcal{O}$. The Euler product decomposition is

$$\zeta_{\Gamma}(s) = \zeta_{\mathrm{H}(\mathbb{C})}(s)^{|k:\mathbb{Q}|} \cdot \prod_{v \notin S} \zeta_{\mathrm{H}(\mathcal{O}_v)}(s) \,.$$

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- The first factor enumerates the *rational* irreducible representations of the group $\mathrm{H}(\mathbb{C})$ and is known as Witten zeta function.
- The factors indexed by $v \notin S$ are representation zeta functions of compact p-adic analytic groups counting irreducible representations with *finite image* (i.e. continuous irreducible representations).

Potent and saturable subgroups

Let G be a connected simply connected semisimple linear algebraic group defined over $\mathbb{Z}$ with Lie algebra $\mathbf{g} = \mathrm{Lie}(\mathrm{G})$. Let k be a number field with ring of integers $\mathcal{O}$ and completion $\mathfrak{o}$ with respect to a prime ideal $\mathfrak{p}$. We set $G = \mathrm{G}(\mathfrak{o})$ and $\mathfrak{g} = \mathbf{g}(\mathfrak{o})$.

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Proposition (Avni, Klopsch, Onn and Voll, 2013)

Let $e = e(\mathfrak{o}, \mathbb{Z}_p)$ be the absolute ramification index of $\mathfrak{o}$. If $m > e \cdot (p-1)^{-1}$, then G^m is saturable. Moreover, if $p > 2$ and $m \geq e \cdot (p-2)^{-1}$, then G^m is potent. If $p = 2$ and $m \geq 2e$, then G^m is potent.

Sheets

Let $\mathcal{L} = \mathbf{g}(\mathbb{C})$ and let $d = \dim_{\mathbb{C}} \mathcal{L}$. We define the locus of constant centralizer dimension $k \leq d$

$$\mathrm{X}^k_{\mathcal{L}}(\mathbb{C}) = \{x \in \mathcal{L} \mid \dim_{\mathbb{C}} \mathrm{C}_{\mathcal{L}}(x) = k\}.$$

and we set

$$f_k = \dim_{\mathbb{C}} \mathrm{X}^k_{\mathcal{L}}(\mathbb{C}),$$

Zeta function as product of geometric progressions

Theorem (MZ)

Let $\mathcal{S} \subseteq \{1, \ldots, d\}$ be the set of all possible dimensions for centralizers in $\mathcal{L}$.

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Theorem (MZ)

Let $\mathcal{S} \subseteq \{1, \dots, d\}$ be the set of all possible dimensions for centralizers in $\mathcal{L}$. Assume that the Killing form on $\mathfrak{g}$ is non-degenerate. Assume further that $\mathfrak{g}$ has smooth and irreducible loci of constant centralizer dimension.

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Theorem (MZ)

Let $\mathcal{S} \subseteq \{1, \ldots, d\}$ be the set of all possible dimensions for centralizers in $\mathcal{L}$. Assume that the Killing form on $\mathfrak{g}$ is non-degenerate. Assume further that $\mathfrak{g}$ has smooth and irreducible loci of constant centralizer dimension. Then for all $m \in \mathbb{N}$ such that G^m is potent and saturable

$$\zeta_{G^m}(s) = q^{d \cdot m} \sum_{I \subseteq \mathcal{S}} \mathrm{g}_{\mathfrak{g},I}(q) \cdot \prod_{i \in I} \frac{q^{f_i - (d-i)\frac{s+2}{2}}}{1 - q^{f_i - (d-i)\frac{s+2}{2}}}.$$

$\zeta_{\mathrm{SL}_4^m(\mathfrak{o})}(s)$

Let $\mathfrak{o}$ be a compact discrete valuation ring of characteristic 0 whose residue field has cardinality q and characteristic not equal to 2. Then, for all $m \in \mathbb{N}$ such that $\mathrm{SL}_4^m(\mathfrak{o})$ is potent and saturable,

$$\zeta_{\mathrm{SL}_4^m(\mathfrak{o})}(s) = q^{15m} \frac{\mathcal{F}(q, q^{-s})}{\mathcal{G}(q, q^{-s})}$$

where

$$
\begin{aligned}
\mathcal{F}(q,t) = {} & qt^{18} - \left(q^7 + q^6 + q^5 + q^4 - q^3 - q^2 - q\right)t^{15} \\
& + \left(q^8 - 2\,q^5 - q^3 + q^2\right)t^{14} \\
& + \left(q^9 + 2\,q^8 + 2\,q^7 - 2\,q^5 - 4\,q^4 - 2\,q^3 - q^2 + 2\,q + 1\right)t^{13} \\
& - \left(q^{10} + q^9 + q^8 - 2\,q^7 - 2\,q^6 - 2\,q^5 + 2\,q^3 + q^2 + q\right)t^{12} \\
& + \left(q^8 + 2\,q^6 + q^4 - q^3 - q^2 - q\right)t^{11} + \left(q^8 + q^7 - 2\,q^4 + q\right)t^{10} \\
& - \left(2\,q^{10} + q^9 + q^8 - q^7 - 3\,q^6 - 2\,q^5 - 3\,q^4 - q^3 + q^2 + q + 2\right)t^9 \\
& + \left(q^9 - 2\,q^6 + q^3 + q^2\right)t^8 - \left(q^9 + q^8 + q^7 - q^6 - 2\,q^4 - q^2\right)t^7 \\
& - \left(q^9 + q^8 + 2\,q^7 - 2\,q^5 - 2\,q^4 - 2\,q^3 + q^2 + q + 1\right)t^6 \\
& + \left(q^{10} + 2\,q^9 - q^8 - 2\,q^7 - 4\,q^6 - 2\,q^5 + 2\,q^3 + 2\,q^2 + q\right)t^5 \\
& + \left(q^8 - q^7 - 2\,q^5 + q^2\right)t^4 + \left(q^9 + q^8 + q^7 - q^6 - q^5 - q^4 - q^3\right)t^3
\end{aligned}
$$

$$
\mathcal{G}(q,t) = q^9\left(1 - qt^3\right)\left(1 - qt^4\right)\left(1 - q^2t^5\right)\left(1 - q^3t^6\right).
$$