OUTER AUTOMORPHISMS OF THE RATIONAL GROUP ALGEBRAS OF FINITE GROUPS

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The talk is based on a joint work with M. Dokuchaev (Univ. of Saõ Paolo).

The group algebras of finite groups over the rationals is a popular area of study. There are many publications discussed various problems concerning rational group algebras.

This work was inspired by the paper by W. Feit and G. Seitz "On finite rational group algebras and related topics", Illinois J. Math. 33(1988), 103 - 131. Apart of a number of interesting results, they proved the following theorem.

Theorem 1. The rational group algebra of a finite group $G \neq \{1\}$ has an outer automorphism.

We generalize this by proving

Theorem 2. The rational group algebra of a finite group $G \neq 1$ has an outer automorphism of order 2, unless G is isomorphic to $SL_2(8)$ or ${}^3D_4(2)$.

In other words, there are only three groups listed above with the group of outer automorphisms of their group algebra over the rationals is of odd order.

The proof uses the classification of finite simple groups (likewise the proof of Feit-Seitz's theorem does).

Notation. $\mathbb{Q}$ is the field of rational numbers, $\mathbb{Q}G$ is the group algebra of a finite group G over $\mathbb{Q}$ and $\operatorname{Out}\mathbb{Q}G$ the group of outer automorphisms of $\mathbb{Q}G$.

The primary observation is

Lemma. Let N be a normal subgroup of a finite group G. If the order of $\operatorname{Out}\mathbb{Q}(G/N)$ is even then so is the order of $\operatorname{Out}\mathbb{Q}G$.

Indeed, $\mathbb{Q}G = \mathbb{Q}(G/N) \oplus R$, where R is the sum of simple components of $\mathbb{Q}G$, non-trivial on N.

This indicate that one first has to examine simple groups. The abelian group case is easy:

Lemma. If G is of prime order then $\operatorname{Out} \mathbb{Q} G$ has even order.

Indeed, if $|G| = 2$ then $\mathbb{Q} G = \mathbb{Q} \oplus \mathbb{Q}$, so $\operatorname{Out} \mathbb{Q} G$ has order 2. If $p > 2$ then $\mathbb{Q} G = \mathbb{Q} \oplus \mathbb{Q}(\varepsilon)$, where ε is a primitive p-root of unity. Then

$$\operatorname{Out} \mathbb{Q} G \cong \operatorname{Gal} \mathbb{Q}(\varepsilon) / \mathbb{Q} .$$

The order of $\operatorname{Gal} \mathbb{Q}(\varepsilon) / \mathbb{Q}$ is $p - 1$, so it is even.

This reflects what we do in general:

given $G \neq SL_2(8), {}^3D_4(2)$, we find either two isomorphic simple components of $\mathbb{Q}G$, or a single one with an outer automorphism of order 2.

For non-abelian simple groups the problem is not trivial.

The following result of Feit and Seitz significally reduces the search:

Lemma. If G has an outer automorphism of order 2 then so is $\mathbb{Q}G$.

The simple groups with no involutory outer automorphism are well known.

Apart from 11 sporadic simple groups, and groups of order ≤ 2, these are:

$Sp_{2n}(q)$ with q even, $2 \neq n \geq 1$, $G_2(q)$, $3 \not| q$, $F_4(q)$, q odd, $E_7(q)$, q even, $E_8(q)$, ${}^3D_4(q)$, q is not a square in all these cases, and ${}^2B_2(q)$, ${}^2G_2(q)$, ${}^2F_4(q)$.

For these groups we need deeper insight into their irreducible character properties.

There are several ways to deal with these groups, but those with explicit character tables are easier.

Let $\mathrm{Irr}\, G$ denote the set of all irreducible characters of G, and $F(G)$ the minimal field containing all $\chi(g)$ for $\chi \in \mathrm{Irr}\, G$ and $g \in G$. Then $F(G)$ coincides with the minimal field containing all $Z(R)$ when R runs over the simple component of $\mathbb{Q}\, G$.

Lemma. If the index $F(G) : \mathbb{Q}$ is even and all characters of G have Schur index at most 2 then $|\operatorname{Out} \mathbb{Q} G|$ is even.

The converse is not true, say, for the symmetric group S_n we have $\mathbb{Q}(G) = \mathbb{Q}$ but $|\operatorname{Out} \mathbb{Q} G|$ is even.

In case $G = Sp_{2n}(q)$ we use induction on n, based on the notion of isolated subgroup. Note that the Schur index of every irreducible representation of G is known to equal 1.

Let H be a subgroup of G. Then H is called isolated if for every cyclic subgroup $C \subset H$ we have $N_G(C)/C_G(C) = N_H(C)/C_H(C)$. In other words, $N_G(C) = N_H(C) \cdot C_G(C)$.

Proposition. If $H \subset G$ is isolated then $\mathbb{Q}(H) \subseteq \mathbb{Q}(G)$.

Lemma. Let $G = Sp_{2n}(q)$ and $H \cong Sp_{2m}(q)$, $m < n$, be a natural subgroup of G.
Then H is isolated.

So we use induction on n to show that $F(G) : \mathbb{Q}$ is even, starting with $SL_2(q)$, $q > 2$, and $SL_8(2)$ for $q = 2$.

Suppose G is not simple. We are lucky that the following is true:

Lemma. If $G = SL_2(8) \times {}^3D_4(2)$ then the order of $\mathrm{Out}\,\mathbb{Q}\,G$ is even.

This implies that it suffices to prove the theorem for groups with a unique normal subgroup N, say, with $1 \neq N \neq G$, and such that $G/N = SL_2(8)$ or ${}^3D_4(2)$.

Then either N is a direct product of non-abelian simple groups or N is an elementary abelian p-group for some prime p.

In this case the strategy is to construct two isomorphic components of $\mathbb{Q}G$. We do this by providing two $\mathbb{Q}$-valued induced characters of Schur index 1 and of the same degree.

I mention some details in case N is abelian. We called $|N|$ large if G/N has at least two regular orbits on N. If N is large then one can construct two non-isomorphic monomial irreducible representations of degree G/N with Schur index 1. These yield two isomorphic components of $\mathbb{Q}G$.

Using a counting argument found by Liebeck (1995), we show that N is large unless, possibly,

$G = SL_2(8)$ and $|N| = 2^6$ or $G/N \cong {}^3D_4(2)$ and $|N| \in \{3^{25}, 2^{26}, 2^{24}\}$.

In these cases G/N has no regular orbit on N. So we need another approach.

We deal with these cases using information from the Brauer character table of G/N.