First-order Group Theory and Branch Groups

John Wilson

jsw13@cam.ac.uk; John.Wilson@maths.ox.ac.uk

Ischia, 31 March 2016

First-order group theory

$(\forall x \forall y \forall z)([x, y, z] = 1)$	G nilp. of class $\leqslant 2$	Yes!
$(\forall x \in G')(\forall z)([x, z] = 1)$	G nilp. of class $\leqslant 2$	No!

$(\forall x_1 \forall x_2 \forall x_3 \forall x_4)(\exists y_1, y_2)([x_1, x_2][x_3, x_4] = [y_1, y_2])$

every element of G' is a commutator

$(\forall x_1 \forall x_2 \exists y)(y \neq x_1 \wedge y \neq x_2)$	$\lvert G \rvert \geqslant 3$	
$(\forall x_1 \forall x_2 \forall x_3 \forall x_4)(\bigvee_{1 \leqslant i < j \leqslant 4} x_i = x_j)$	$\lvert G \rvert \leqslant 3$	
$(\forall x)(x^6 = 1 \to x = 1)$	no elements of order $2, 3$	
$g^4 = 1 \wedge g^2 \neq 1$	g has order 4	
$(\exists n)(g^n = 1)$	g has finite order	No!
$(\forall x \in G')(x^7 = 1)$	G' has exponent dividing 7	No!

Some finite axiomatizations

(1) {groups of order $\leqslant n$}, {groups of order $\geqslant n$}, {groups with no elements of order n}

(2) Let $H = \{h_1, \ldots, h_n\}$ be finite, $h_i h_j = h_{\mu(i,j)}$

$\theta_H(x_1, \ldots, x_n)$: $(\bigwedge_{i \neq j}(x_i \neq x_j) \wedge \bigwedge_{i,j}(x_i x_j = x_{\mu(i,j)}))$
ϕ_H: $(\exists x_1 \cdots \exists x_n)\ \theta_H(x_1, \ldots, x_n)$
ψ_H: $(\exists x_1 \cdots \exists x_n)(\forall y)(\theta_H(x_1, \ldots, x_n) \wedge (\bigvee_i y = x_i))$

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(3) Soluble groups: defined by 'no $g \neq 1$ is a prod. of commutators $[g^h, g^k]$'; that is, ρ_n holds $\forall n$

$$\rho_n: (\forall g \forall x_1 \ldots \forall x_n \forall y_1 \ldots \forall y_n)(g = 1 \vee g \neq [g^{x_1}, g^{y_1}] \ldots [g^{x_n}, g^{y_n}]).$$

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Theorem (JSW) A finite group is soluble iff it satisfies ρ_{56}.

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- Centralizers of definable sets are definable:
  Say $S = \{s \mid \varphi(s)\}$; then $C_G(S) = \{t \mid \forall g(\varphi(g) \to [g, t] = 1)\}$

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So $\exists$ f.o. formula ω_h with $\omega_h(g)$ iff $g \in \mathrm{C}_G\mathrm{C}_G(W_h)$
- $\delta(x, y)$: $\delta(h_1, h_2)$ iff $\mathrm{C}^2_G(W_{h_1}) = \mathrm{C}^2_G(W_{h_2})$
- $\exists$ $\beta(x)$: $\beta(h)$ iff $\mathrm{C}^2_G(W_h)$ commutes with its distinct conjugates

Interpretations: an example

K a field, T the mult. group $K \setminus \{0\}$.

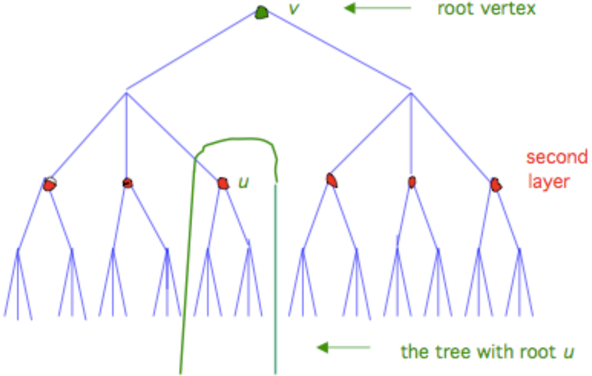
$$G = \left\{ \begin{pmatrix} 1 & x \\ 0 & t \end{pmatrix} \mid x \in K, t \in T \right\}.$$

Write (x, t) for above matrix,
$A = \{(x, 1) \mid x \in K\} \cong K_+$ and $H = \{(0, t) \mid t \in T\} \cong T$.
So $A \triangleleft G$, $G = A \rtimes H$. Fix $e = (1, 1) \in A$.
$A = \{k \mid (\forall g)\,[k^g, k] = 1\}$ definable in G, and
$H = \{g \mid ge = eg\} = \mathrm{C}_G(e)$ definable (with parameter e).
For a, b in A define
$a + b = ab$,

$$a * b = \begin{cases} 1 & \text{if } a \text{ or } b = 1 \\ a^g & \text{if not, where } b = e^g \text{ with } g \in G. \end{cases}$$

A becomes a field isomorphic to K.
The set A and the operations on A are definable in G. An interpretation (with parameter e) of the field K in the group G.

The rooted tree of type $(2, 3, 2, 3, \ldots)$



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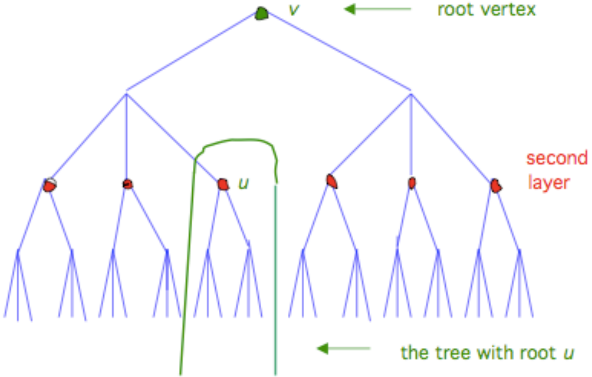

Let G act faithfully on T fixing v.

Second layer L_2 is a union of G-orbits

$\mathrm{rst}_G(u)$ – elements moving only vertices in T_u

$\mathrm{rst}_G(2) = \langle \mathrm{rst}_G(w) \mid w \in$ 2nd layer$\rangle$, dir. product of six conj. subgroups

Basal subgroup B: the distinct conjs. of B generate their dir. product

Fix $(m_n)_{n\geqslant 0}$, a sequence of integers $m_n \geqslant 2$.

The *rooted tree* T of type (m_n) has a root vertex v_0 of valency m_0. Each vertex of distance $n \geqslant 1$ from v_0 has valency $m_n + 1$.
nth layer L_n: all vertices u at distance n from v_0.
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Let G act faithfully on T.
$\mathrm{rst}_G(u) = \{g \mid g \text{ fixes each vertex outside } T_u\}$.
$\mathrm{rst}_G(n) = \langle \mathrm{rst}_G(u) \mid u \in L_n \rangle$.

G acts as a *branch group* on T if for each n,

- G acts transitively on L_n,
- $\mathrm{rst}_G(n)$ has finite index in G.

Examples, motivation

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(2) Numerous other important examples

(3) Many examples G are just infinite (JI); i.e., infinite and G/K finite whenever $1 \neq K \lhd G$.

G is called hereditarily just infinite (HJI) if every subgroup of finite index is JI.

(JSW, 1972) If G is JI and not virtually abelian then either

- $G \leqslant_f H \operatorname{wr} \operatorname{Sym}(m)$ for some m where H is HJI, or
- G is a branch group.

G is **virtually abelian (vA)** if G has an abelian subgroup of finite index.

Fundamental Lemma (Grigorchuk). If G is a branch group on T and $1 \neq K \lhd G$ then $\mathrm{rst}_G(n)' \leqslant K$ for some n.

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Lemma. (Grigorchuk & JSW, 2002) Branch groups have no non-triv. vA normal subgroups.

Definition. G is Boolean if $G \neq 1$ and
- G/K is vA (virtually abelian) whenever $1 < K \leqslant G$;
- G has no non-trivial vA normal subgroups.

So branch groups are Boolean.

Structure lattice

Assume G is Boolean.

$\mathbf{L}(G) = \{H \leqslant G \mid |G : \mathrm{N}_G(H)| \text{ finite}\}$

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The lattice operations in $\mathbf{L}(G)$ induce well-defined join and meet operations $\vee$, $\wedge$ in

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$\mathcal{L}$ is the structure lattice of G; greatest and least elements $[G]$ and $[1] = \{1\}$.
$\mathcal{L}$ is a Boolean lattice: complemented with distributive laws:
$a \vee (b_1 \wedge b_2) = (a \vee b_1) \wedge (a \vee b_2), \ldots$

Basal subgroups

Definition. B is a basal subgroup of G if $B \in \mathbf{L}(G)$ and $\langle B^g \mid g \in G \rangle$ is the **direct** product of the distinct conjugates of B.

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Examples:

(1) the natural direct factors of the base group of $H \operatorname{wr} \operatorname{Sym}(m)$;
(2) restricted stabilizers of vertices in branch groups.

Structure graph

Branch groups G can have branch actions on essentially different maximal trees. These actions are encoded in the structure graph $\Gamma(G)$:

$$\Gamma(G) = \{[B] \mid B \text{ basal in } G\} \quad (\subseteq \mathcal{L}(G)\ ).$$

edges are the pairs (a, b) with b maximal in $\{c \mid c \in \Gamma(G), c < a\}$.

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The tree on which G acts embeds equivariantly in the structure graph; often the embedding is an equivariant IM of trees.

In this case G 'knows' its tree: can find the tree within G.

New description of structure graph

For $Y \subseteq G$ write $\mathrm{C}_G^2(Y)$ for $\mathrm{C}_G(\mathrm{C}_G(Y))$, etc. So $Y \subseteq \mathrm{C}_G^2(Y)$, $\mathrm{C}_G^3(Y) = \mathrm{C}_G(Y)$.
$H \in \mathbf{L}(G)$ is C^2-closed if $H = \mathrm{C}_G^2(H)$.

Lemma Let G be a branch group.

(a) If $H_1, H_2 \in \mathbf{L}(G)$ have same centralizer then $\mathrm{C}_G^2(H_1) = \mathrm{C}_G^2(H_2)$.

(b) B basal $\Rightarrow \mathrm{C}_G^2(B)$ basal.

(c) $B_1 < B_2$ basal, C^2-closed $\Rightarrow \mathrm{N}_G(B_1) < \mathrm{N}_G(B_2)$.

The graph $\mathcal{B}(G)$ has

- vertices the non-trivial C^2-closed basal subgroups,
- edge between vertices if one is a maximal proper C^2-closed basal subgroup of the other.

G acts on $\mathcal{B}(G)$ by conjugation.

Lemma G branch, on tree T, and v a vertex. Then $\mathrm{C}^2_G(\mathrm{rst}_G(v)) = \mathrm{rst}_G(v)$, so $\mathrm{rst}(v) \in \mathcal{B}(G)$.

Proof.

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Theorem. G branch, acting on T.

(a) $B \mapsto [B]$ is a G-equivariant IM $\mathcal{B}(G) \to \Gamma(G)$.

(b) $v \mapsto \mathrm{rst}_G(v)$ is a G-equivt. order-preserving injective map $T \to \mathcal{B}(G)$.

Properties of $\mathcal{B} = \mathcal{B}(G)$ for branch G:

- G is the only vertex fixed in the G-action on $\mathcal{B}$
- the orbit $O(B)$ of each vertex B is finite
- each vertex B is connected to vertex G by a finite path; all simple such paths have length $\leqslant \log_2(|O(B)|)$
- $\forall\ B \in \mathcal{B}\ \ \exists$ branch action for which B is the restricted stabilizer of a vertex
- if $\mathcal{B}$ is a tree then G acts on it as a branch group.

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Questions about $\mathcal{B} = \mathcal{B}(G)$

- finite valency?
- can there be exactly $\aleph_0$ maximal trees?

Maximal trees

Corollary. G branch. The foll. are equivt.:

(i) $\exists$ a unique max. tree up to G-equivariant IM on which G acts as a branch group;

(ii) $\mathcal{B}(G)$ is a tree;

(iii) $\forall\ B,\ B_1,\ B_2 \in \mathcal{B}(G)$ with $B \leqslant B_1$ and $B \leqslant B_2$, either $B_1 \leqslant B_2$ or $B_2 \leqslant B_1$.

Now G is a branch group.

Recall the foll. definition. For each $h \in G$

$$V_h = \{[h^{-1}, h^{k_1}][h^{-1}, h^{k_2}][h^{-1}, h^{k_3}] \mid k_1, k_2, k_3 \in G\},$$
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Key Proposition. $\forall\, B \in \mathcal{B}(G)$, $\exists\, h \in G$ with $B = \mathrm{C}^2_G(W_h)$.

Proof uses (among other things) the result of Hardy, Abért: branch groups satisfy no group laws. In particular, if $u \in T$ then $\exists\, x, y \in \mathrm{rst}_G(u)$ with $(xy)^2 \neq y^2x^2$.

Interpretation in branch groups

Theorem (JSW, 2015). There are first-order formulae τ, $\beta(x)$, $\delta(x, y)$ s.t. the following holds for each branch group G:

(a) G has a branch action on a unique maximal tree up to G-equivariant IM iff $G \models \tau$;

(b) $S = \{x \mid \beta(x)\}$ is a union of conj. classes, so G acts on it by conjugation;

(c) the relation on S defined by $\delta(x, y)$ is a G-invariant preorder. So $Q = S/\sim$, where $\sim$ is the equiv. relation defined by $\delta(x, y) \wedge \delta(y, x)$, is a poset on which G acts;

(d) Q is G-equivariantly isom. as poset to structure graph¿

When G has a branch action on a unique maximal tree T, this represents T as quotient of a definable subset of G modulo a definable equivalence relation. A parameter-free interpretation for T, and for the action on T.

The results give a very weak sort of axiomatization of the class of branch groups.

Similar ideas (with sets like the sets W_h) apply in other contexts.

Current joint work with Andrew Glass:

$\mathrm{Aut}_O(\Omega)$ = group of order-preserving automs. of ordered set Ω.

Theorem. If $\mathrm{Aut}_O(\Omega)$ is transitive on Ω and $\mathrm{Aut}_O(\Omega)$, $\mathrm{Aut}_O(\mathbb{R})$ satisfy the same first-order sentences (in a slightly extended language) then Ω, $\mathbb{R}$ are isomorphic as ordered sets.

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- ideas of first-order group theory
- definition of a branch group

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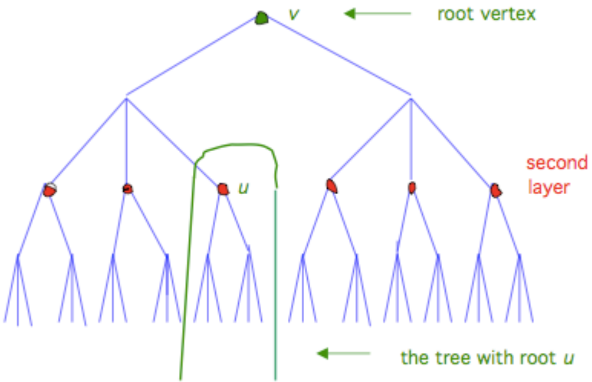

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- ideas of first-order group theory
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  and basal subgroups
- the restricted stabilizers of vertices are basal subgroups B with $\mathrm{C}^2(B) = B$
- if G acts as a branch group on a 'unique tree' then this tree is essentially the set of basal subgroups B with $\mathrm{C}^2(B) = B$
- there's a first-order sentence deciding whether the above holds

  and then G 'knows' its tree in the sense of first-order group theory