On the Hirsch-Plotkin radical of stability groups

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Ischia Group Theory 2016

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1. Stability groups of subspace series.
2. Characterisation of elts that stabilise a finite subseries.
3. A counterexample.
4. A strong local nilpotence property.

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Definition. Let V be a vector space over a field $\mathbb{F}$. A set $\mathcal{L}$, consisting of subspaces of V, is said to be a series in V if

(1) Both 0 and V belong to $\mathcal{L}$.
(2) The set $\mathcal{L}$ is linearly ordered with respect to inclusion.
(3) For every $\mathcal{F} \subseteq \mathcal{L}$ both $\bigcap\{W : W \in \mathcal{F}\}$ and $\bigcup\{W : W \in \mathcal{F}\}$ are in $\mathcal{L}$.

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Definition. A subseries of $\mathcal{L}$ is a subset of $\mathcal{L}$ which is a series.

Remark. Let $0 \neq b \in V$ and let

$$T = \bigcap \{U \in \mathcal{L} : v \in U\}, \;\; B = \bigcup \{U \in \mathcal{L} : v \notin U\}.$$

Then (B, T) is a jump and $v \in T \setminus B$. Thus if $\mathcal{J}$ is the collection of all the jumps of $\mathcal{L}$ then

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Defintion. Let $\mathcal{L}$ be a series in a vector space V. We say that an element in $g \in \mathrm{GL}(V)$ stabilises $\mathcal{L}$ if $T(g-1) \leq B$ for all jumps (B, T) of $\mathcal{L}$.

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Example(T 2015). Let $V = \bigoplus_{i=0}^{\infty} \mathbb{F}u_i + \prod_{j=0}^{\infty} \mathbb{F}w_j$ and $\mathcal{L} = \{V_i : i \in \mathbb{N}\} \cup \{0\}$ where $V_n = \bigoplus_{i=n}^{\infty} \mathbb{F}u_i + \prod_{j=n}^{\infty} \mathbb{F}w_j$.

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Let $R = \{y \in \text{End}(V) : y(V_i) \subseteq V_{i+1},$ for all $i \in \mathbb{N}\}$. Then $S(\mathcal{L}) = 1 + R$. Let $x \in R$ such that $x(u_i) = w_{i+1}$ and $x(W) = 0$. Then $1 + x \in \text{S}(\mathcal{L})$ but $1 + x \notin \text{F}(\mathcal{L})$. It remains to show that $1 + x \in \text{HP}(\mathcal{L})$.

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Let $R=\{y\in \mathrm{End}(V):\ y(V_i)\subseteq V_{i+1},$ for all $i\in\mathbb{N}\}$. Then $S(\mathcal{L})=1+R$. Let $x\in R$ such that $x(u_i)=w_{i+1}$ and $x(W)=0$. Then $1+x\in \mathrm{S}(\mathcal{L})$ but $1+x\notin \mathrm{F}(\mathcal{L})$. It remains to show that $1+x\in \mathrm{HP}(\mathcal{L})$.

Lemma. Let $y\in R$. There exists a positive integer m such that $w_jy\in W$ for all $j\geq m$. It follows that $wy\in W$ for all w$\in W\cap V_m$.

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Lemma. Let $y \in R$. There exists a positive integer m such that $w_j y \in W$ for all $j \geq m$. It follows that $wy \in W$ for all w$\in W \cap V_m$.

Corollary. $1 + x \in \mathrm{HP}(\mathcal{L})$.

4. A strong local nilpotence property

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Theorem(T 2015). For each finitely generated subgroup $H = \langle h_1, \ldots, h_n \rangle$ of HP$(\mathcal{L})$, there exists a positive integer $d = d(H)$ such that $[V,_d H] = 0$.