Generalized Hultman Numbers, and New Generalized Hultman Numbers and it's connection to Generalized commuting probability

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Generalized Commuting Probability

Let $\pi = \langle \pi_1 \;\; \pi_2 \;\; ... \;\; \pi_n \rangle$ be a permutation from S_n, and G is a finite group.

Then $Pr_\pi(G)$ defined as the probability of

$$Pr(a_1 a_2 \cdots a_n = a_{\pi_1} a_{\pi_2} \cdots a_{\pi_{n-1}} a_{\pi_n})$$

in G.

Notice that $Pr_{\langle 2 \;\; 1 \rangle}(G)$ is just the commuting probability of G.

$$Pr^t(G) := Pr_{\langle t \;\; t-1 \ldots 2 \;\; 1\rangle}(G)$$

$$Pr_\pi(G) = Pr^t(G)$$

where t is a non-negative integer number such that $n-t+1$ is the number of alternating cycles in the Hultman decomposition of the cycle graph of the permutation $\pi \in S_n$.

Two directions of generalization

Direction 1: Let G be a finite group, and π be a signed-permutation in B_n. Then,

$$Pr_\pi(G) := Pr(a_1 a_2 \cdots a_n = a_{|\pi_1|}^{\epsilon_1(\pi)} a_{|\pi_2|}^{\epsilon_2(\pi)} \cdots a_{|\pi_n|}^{\epsilon_n(\pi)})$$

where for every $1 \leq i \leq n$, $\epsilon_i \in \{1, -1\}$ is determined as follows:

- In case $\pi_i > 0$, $\epsilon_i(\pi) = 1$.
- In case $\pi_i < 0$, $\epsilon_i(\pi) = -1$

$\pi \in B_n$ is positive, in case $\epsilon_i(\pi) = 1$, for every $1 \leq i \leq n$. Otherwise π is non-positive.

$$Pr^{-t}(G) := Pr_{\langle -1 \ \ -2 \cdots -(t-1) \ \ -t \rangle}(G) =$$

$$= Pr(a_1 a_2 \cdots a_t = a_1^{-1} a_2^{-1} \cdots a_t^{-1}) =$$

$$= Pr(a_1^2 \cdot a_2^2 \cdots a_t^2 = 1)$$

For every non-positive π,

$$Pr_\pi(G) = Pr^{-t}(G)$$

where t is a non-negative integer number such that $n-t+1$ is the number of generalized alternating cycles in the Generalized Hultman decomposition of the cycle graph of the signed-permutation $\pi \in B_n$.

There are two interesting cases:

Ambivalent Groups: In case of G is a finite ambivalent group, $Pr^{-2k}(G) = Pr^{2k}(G)$, for every integer $k \leq 0$. Therefore,

$$Pr_{\pi}(G) = Pr^{\theta}(G)$$

if and only if π and θ have the same number of generalized alternating cycles in the Generalized Hultman decomposition, without depending on weather π or θ is positive or non-positive.

Groups of odd order: In case of G is a finite group, which order is odd, $Pr^{-k}(G) = \frac{1}{|G|}$, for every integer $k \geq 1$.

Direction 2: Let G be a finite group, b be an involution in G, and π be a signed-permutation in B_n. Then,

$$Pr_{\pi,b}(G) := Pr(a_1 a_2 \cdots a_n = a_{|\pi_1|}^{\epsilon_1(\pi)} a_{|\pi_2|}^{\epsilon_2(\pi)} \cdots a_{|\pi_n|}^{\epsilon_n(\pi)})$$

where for every $1 \leq i \leq n$, $\epsilon_i \in \{1, b\}$ is determined as follows:

- In case $\pi_i > 0$, $\epsilon_i(\pi) = 1$.

- In case $\pi_i < 0$, $\epsilon_i(\pi) = b$

For every non-negative integers k, l:

$$Pr_b^{2k,0}(G) := Pr^{2k}(G)$$

$$Pr_b^{2k,l}(G) :=$$

$$Pr(\prod_{i=1}^{2k+2l} a_i = \prod_{i=2k}^{1} a_i \cdot \prod_{i=1}^{l} a_{2k+2i-1}^{b} \cdot a_{2k+2i}) =$$

$$= Pr(\prod_{i=1}^{k} [a_{2i-1}, a_{2i}] = \prod_{i=1}^{2l} b^{a_{2k+i}})$$

$Pr_{b_1}^{2k,l}(G) = Pr_{b_2}^{2k,l}(G)$, for every non-negative integers k, l, in case b_1 and b_2 are conjugate involutions.

New Generalized Hultman Decomposition:

Let $\pi \in B_n$. then look at the set H_n of $2n+2$ vertices named by

$$H_n = \{0, 1, ..., n\}$$

$(i+) := (i+1) \bmod (n+1)$,
$(i-) := (i-1) \bmod (n+1)$.

There are black-edges connecting $i \dashrightarrow i+$, for every $i \in H_n$.

There are gray-edges connecting $\pi(i) \to \pi(i-)$ for every $i \in H_n$.

The gray-edges are labeled by $+$ or $-$ as follows:

- In case both π and $\pi(i-)$ are positive or negative:
  The gray-edge connecting $\pi(i)$ to $\pi(i-)$ is labeled by $+$.

- In case only $\pi(i)$ or $\pi(i-)$ is positive:
  The gray-edge connecting $\pi(i)$ to $\pi(i-)$ is labeled by $-$.

The cycle graph $Gr(\phi)$ of a signed permutation $\pi \in B_n$ is the bi-colored directed labeled graph.

- An "alternating negative cycle" is a cycle where black edges are followed by gray alternately, and there are odd number of edges which are labeled by $-$.

- An "alternating positive cycle" is a cycle where a black edge followed by a gray alternately, and there are even number of edges which are labeled by $-$.

- Let $s^+(\pi)$ be the number of the alternating positive cycles of $Gr(\pi)$.

- Let $s^-(\pi)$ be the number of the alternating negative cycles of $Gr(\pi)$.

- Let $s(\pi) = s^+(\pi) + \frac{s^-(\pi)}{2}$.

Theorem: Let G be a finite group, $b \in G$ an involution, $\pi \in B_n$, then:

$$Pr_{\pi,b}(G) = Pr_b^{2k,l}(G),$$

where

$$l = \frac{s^-(\pi)}{2},$$

and

$$2k = n - s^+(\pi) - s^-(\pi) + 1.$$

and therefore,

$$2k + l = n - s(\pi) + 1.$$