# Measuring the length of a finite group

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Various notions related to the length of a finite group have been introduced in the context of the Restricted Burnside Problem. Therefore I would like to start with a few words on the RBP. There are several equivalent ways to formulate it.

1. Let $m, e$ be positive integers. Is the order of any $m$-generated finite group of exponent $e$ bounded in terms of $m$ and $e$ only?
2. Is every residually finite group of finite exponent locally finite?

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The Lie-theoretical part of the RBP was solved by Zelmanov in the late eighties. We will now discuss in some detail the contribution of Hall and Higman.

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If $G$ is $p$-soluble, then
(a) $l_p(G) \leq d_p(G)$,
(b) $l_p(G) \leq e_p(G)$ if $p$ is not a Fermat prime,
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Hall and Higman proved this for $p$ odd. The case $p = 2$ was proved by Bryukhanova in 1979 and 1981.

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# A generalization of The Restricted Burnside Problem

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This fails if the assumption of residual finiteness is dropped.
If $G$ is not residually finite, $G'$ need not even be periodic! (Adian, Deryabina – Kozhevnikov)

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Another natural question is for which verbal subgroups of $G$ a similar phenomenon holds.

Given a group-word $w$ in variables $x_1, \dots, x_t$ we think of it primarily as a function of $t$ variables defined on any given group $G$. We denote by $w(G)$ the verbal subgroup of $G$ generated by the values of $w$.

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PROBLEM: *Let $n$ be a positive integer and $w$ a word. Assume that $G$ is a residually finite group such that any $w$-value in $G$ has order dividing $n$. Does it follow that the verbal subgroup $w(G)$ is locally finite?*

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The solution to the RBP shows that the answer to Problem is positive if $w = x$.

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Particular examples of multilinear commutators are the derived words, defined by the equations:

$$\delta_0(x) = x,$$

$$\delta_k(x_1, \dots, x_{2^k}) = [\delta_{k-1}(x_1, \dots, x_{2^{k-1}}), \delta_{k-1}(x_{2^{k-1}+1} \dots, x_{2^k})],$$

and the lower central words:

$$\gamma_1(x) = x,$$

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The nonsoluble length $\lambda(G)$ of $G$ does not exceed $2L_2 + 1$, where $L_2$ is the maximum 2-length of soluble subgroups of $G$.

For $p \neq 2$, the non-$p$-soluble length $\lambda_p(G)$ of $G$ does not exceed the maximum $p$-length of $p$-soluble subgroups of $G$.

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Unlike the situation considered by Hall and Higman, the proofs of these results are pretty complicated. The Schreier conjecture is used in the proofs again (so the proofs depend on the classification of finite simple groups).

From this we deduce the following surprizing theorem.

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The following theorem was recently proved by Casolo, Jabara, and Spiga.

**Theorem**

*Let $G = AB$ be a finite soluble group factorised by its proper subgroups $A$ and $B$ with $\gcd(|A|, |B|) = 1$. Then $h(G) \leq h(A) + h(B) + 4d(B) - 1$. Moreover, if $|B|$ is odd, then $h(G) \leq h(A) + h(B) + 2d(B) - 1$, and if $B$ is nilpotent, then $h(G) \leq h(A) + 2d(B)$.*

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**Theorem**

*Suppose that a finite group $G = AB$ admits a factorization by two subgroups $A$, $B$ of coprime orders. Then the nonsoluble length $\lambda(G)$ of $G$ is bounded in terms of the generalized Fitting heights $h^*(A)$ and $h^*(B)$ of the factors. More precisely, $\lambda(G) \leq 2^{h^*(A)+h^*(B)} - 1$.*

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**Theorem**

*Suppose that a finite group $G = AB$ admits a factorization by two subgroups $A$, $B$ of coprime orders, of which $B$ is soluble of derived length $d$. Then the generalized Fitting height $h^*(G)$ of $G$ is bounded in terms of $d$ and $h^*(A)$.*

# LENGTH OF GROUPS WITH AUTOMORPHISMS

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In a recent work with Khukhro we proved results for the case where $G$ is nonsoluble.

### Theorem

*Suppose that a finite group $G$ admits a soluble group of automorphisms $A$ of coprime order. Then its generalized Fitting height $h^*(G)$ is bounded in terms of the generalized Fitting height $h^*(C_G(A))$ of the fixed-point subgroup $C_G(A)$ and the number of prime factors of $|A|$ counting multiplicities.*

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It would be interesting to see if this bound can be improved (perhaps it should be linear?).

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In the case of $w = [x, y]$ the result is known to hold also for $p = 2$.