Algebras and regular subgroups

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Ischia Group Theory 2016

Let $\mathbb{F}$ be any field.

We identify the affine group $\mathrm{AGL}_n(\mathbb{F})$ with the subgroup of $\mathrm{GL}_{n+1}(\mathbb{F})$ consisting of the matrices having $(1, 0, \ldots, 0)^{\mathrm{T}}$ as first column.
So, $\mathrm{AGL}_n(\mathbb{F})$ acts on the right on the set $\mathcal{A} = \{(1, v) : v \in \mathbb{F}^n\}$ of affine points.

Definition

A subgroup (or a subset) R of $\mathrm{AGL}_n(\mathbb{F})$ is called regular if it acts regularly on $\mathcal{A}$, namely if, for every $v \in \mathbb{F}^n$, there exists a unique element in R having $(1, v)$ as first row.

Literature

The problem of classifying the regular subgroups of $\mathrm{AGL}_n(\mathbb{F})$ attracted the interest of many authors:

Caranti, Dalla Volta, Sala (2006); Catino, Rizzo (2009); Catino, Colazzo, Stefanelli (2015-2016); Childs (2015).

These authors use concepts as radical rings, braces and Hopf structures.
We prefer to adopt an approach based only on linear algebra.
On the other hand this problem is also connected to the classification of nilpotent algebras:

Childs (2005); Poonen (2008); De Graaf (2010).

We write every element r of a regular subgroup R of $\mathrm{AGL}_n(\mathbb{F})$ as

$$r = \begin{pmatrix} 1 & v \\ 0 & \pi(r) \end{pmatrix} = \begin{pmatrix} 1 & v \\ 0 & I_n + \delta_R(v) \end{pmatrix} = \mu_R(v),$$

where

$$\begin{aligned} \pi &: \mathrm{AGL}_n(\mathbb{F}) \to \mathrm{GL}_n(\mathbb{F}), \\ \mu_R &: \mathbb{F}^n \to \mathrm{AGL}_n(\mathbb{F}), \\ \delta_R &: \mathbb{F}^n \to \mathrm{Mat}_n(\mathbb{F}). \end{aligned}$$

Example

The translation subgroup

$$\mathcal{T} = \left\{ \left(\begin{array}{c|c} 1 & v \\ \hline 0 & I_n \end{array} \right) : v \in \mathbb{F}^n \right\}$$

is an example of regular subgroup ($\delta_{\mathcal{T}} = 0$).

LEMMA

Let R be a regular subset of $\mathrm{AGL}_n(\mathbb{F})$ containing I_{n+1} and such that δ_R is additive. Then R is a subgroup, if and only if:

$$\delta_R(v\delta_R(w)) = \delta_R(v)\delta_R(w), \qquad \text{for all } v, w \in \mathbb{F}^n.$$

Also, given $v, w \in \mathbb{F}^n$,

COMMUTATIVITY

$$\mu_R(v)\mu_R(w) = \mu_R(w)\mu_R(v) \quad \Leftrightarrow \quad v\delta_R(w) = w\delta_R(v).$$

Theorem

Let R be regular subgroup of $\mathrm{AGL}_n(\mathbb{F})$. Then the following holds:

(a) *the center $\mathbf{Z}(R)$ of R is unipotent;*

(b) *if δ_R is linear, then R is unipotent;*

(c) *if R is abelian, then δ_R is linear.*

Example

Let $n \geq 4$ and $\mathbb{F} = \mathbb{F}_p$ (p odd). Take the matrix $A = \mathrm{diag}(A_3, I_{n-4})$ of $\mathrm{GL}_{n-1}(\mathbb{F}_p)$, where $A_3 = \begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$. Then A has order p, is orthogonal and its minimum polynomial has degree 3. The subset R of order p^4, defined by

$$R = \left\{ \begin{pmatrix} 1 & v & h + \frac{vJv^{\mathrm{T}}}{2} \\ 0 & A^h & A^h J v^{\mathrm{T}} \\ 0 & 0 & 1 \end{pmatrix} : v \in {\mathbb{F}_p}^{n-1}, h \in \mathbb{F}_p \right\},$$

is a unipotent regular subgroup of $\mathrm{AGL}_n(\mathbb{F}_p)$, such that $R \cap \mathfrak{T} = \{1\}$.
Note that δ_R is not linear.

DEFINITION

An $\mathbb{F}$-algebra $\mathcal{L}$ with 1 is called split local if $\mathcal{L}/\mathbf{J}(\mathcal{L})$ is isomorphic to $\mathbb{F}$, where $\mathbf{J}(\mathcal{L})$ denotes the Jacobson radical of $\mathcal{L}$.

In particular $\mathcal{L} = \mathbb{F}\,1 + \mathbf{J}(\mathcal{L})$, where $\mathbb{F}\,1 = \{\alpha 1_{\mathcal{L}} : \alpha \in \mathbb{F}\}$.

We say that $\mathcal{L}$ is finite dimensional if $\mathbf{J}(\mathcal{L})$, viewed as an $\mathbb{F}$-module, has finite dimension.

Theorem

Let $\mathcal{L}$ be a finite dimensional split local $\mathbb{F}$-algebra. Then, with respect to the product in $\mathcal{L}$, the subset

$$R = 1 + \mathbf{J}(\mathcal{L}) = \{1 + v : v \in \mathbf{J}(\mathcal{L})\}$$

is a group, isomorphic to a regular subgroup of $\mathrm{AGL}_n(\mathbb{F})$ for which δ_R is linear, $n = \dim_{\mathbb{F}}(\mathbf{J}(\mathcal{L}))$.

Conversely we have the following result.

Theorem

Let R be a regular subgroup of $\mathrm{AGL}_n(\mathbb{F})$. Set

$$V = R - I_{n+1} = \left\{ \begin{pmatrix} 0 & v \\ 0 & \delta_R(v) \end{pmatrix} \mid v \in \mathbb{F}^n \right\},$$

$$\mathcal{L}_R = \mathbb{F} I_{n+1} + V.$$

Then $\mathcal{L}_R$ is a split local subalgebra of $\mathrm{Mat}_{n+1}(\mathbb{F})$, with $\mathbf{J}(\mathcal{L}_R) = V$, if and only if δ_R is linear.

Our classification of the regular subgroups of $\mathrm{AGL}_n(\mathbb{F})$ is based on the following proposition (see Caranti, Dalla Volta, Sala (2006)):

Proposition

Assume that R_1, R_2 are regular subgroups of $\mathrm{AGL}_n(\mathbb{F})$ such that δ_{R_1} and δ_{R_2} are linear maps. Then

R_1 and R_2 are conjugate in $\mathrm{AGL}_n(\mathbb{F})$

if and only if

the algebras $\mathcal{L}_{R_1}$ and $\mathcal{L}_{R_2}$ are isomorphic.

Theorem

Let R be a regular subgroup of $\mathrm{AGL}_n(\mathbb{F})$ and $1 \neq z$ be an element of the center $\mathbf{Z}(R)$ of R.
Then, up to conjugation of R under $\mathrm{AGL}_n(\mathbb{F})$, we may suppose that $z = J_z$, where

$$J_z = \mathrm{diag}\left(J_{m_1}, \ldots, J_{m_k}\right)$$

is the Jordan form of z having Jordan blocks J_{m_i} associated to the eigenvalue 1, of respective sizes

$$m_i \geq m_{i+1}$$

for all $i \geq 1$.

Our method

We restrict our attention to regular subgroups U of $\mathrm{AGL}_n(\mathbb{F})$ such that the map δ_U is linear.

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Our classification is obtained working inside $\mathbf{C}_{\mathrm{AGL}_n(\mathbb{F})}(z)$ using the following parameters d, r and k, considered in this order.
Let H be any unipotent subgroup of $\mathrm{AGL}_n(\mathbb{F})$.

$$\begin{aligned} d(H) &= \max\{\deg \min_{\mathbb{F}}(h - I_{n+1}) \mid h \in H\}; \\ r(H) &= \max\{\mathrm{rk}(h - I_{n+1}) \mid h \in H\}; \\ k(H) &= \dim_{\mathbb{F}}\{w \in \mathbb{F}^n : w\pi(h) = w, \forall\, h \in H\}. \end{aligned}$$

Lemma

Let U be a unipotent regular subgroup of $\mathrm{AGL}_n(\mathbb{F})$. If $d(\mathbf{Z}(U)) = n+1$ then, up to conjugation, $U = \mathbf{C}_{\mathrm{AGL}_n(\mathbb{F})}(J_{n+1})$. Moreover U is abelian.

Lemma

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Lemma

Let U be a regular subgroup of $\mathrm{AGL}_n(\mathbb{F})$, $n \geq 2$. If $d(\mathbf{Z}(U)) = n$ then, up to conjugation, for some fixed $\alpha \in \mathbb{F}$:

$$U = R_\alpha = \begin{pmatrix} 1 & x_1 & x_2 & \dots & x_{n-2} & x_{n-1} & x_n \\ 0 & 1 & x_1 & \dots & x_{n-3} & 0 & x_{n-2} \\ 0 & 0 & 1 & \dots & x_{n-4} & 0 & x_{n-3} \\ \vdots & & & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & x_1 \\ 0 & 0 & 0 & \dots & 0 & 1 & \alpha x_{n-1} \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}.$$

In particular U is abelian and $r(U) = n-1$. Furthermore, R_0 and R_α are not conjugate for any $\alpha \neq 0$.

If $n \geq 4$ is even, then R_α is conjugate to R_1 for any $\alpha \neq 0$ and an epimorphism $\Psi : \mathbb{F}[t_1, t_2] \to \mathbb{F} I_{n+1} + R_\alpha$ is obtained setting

$$\Psi(t_1) = \alpha X_1, \qquad \Psi(t_2) = \alpha^{\frac{n-2}{2}} X_{n-1}.$$

In this case $\mathrm{Ker}\,(\Psi) = \langle {t_1}^{n-1} - {t_2}^2, t_1 t_2 \rangle$.

If n is odd, R_α and R_β $(\alpha, \beta \in \mathbb{F}^*)$ are conjugate if and only if β/α is a square in $\mathbb{F}^*$.

Our method: an example

Lemma

Let U be a regular subgroup of $\mathrm{AGL}_3(\mathbb{F})$.
If $d(\mathbf{Z}(U)) = r(\mathbf{Z}(U)) = 2$, then U is abelian, conjugate to

$$\begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & 0 & x_2 \\ 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $\operatorname{char} \mathbb{F} = 2$.

Proof.

We may assume
$$z = \mathrm{diag}(J_2, J_2) \in \mathbf{Z}(U),$$
whence $\delta(v_1) = E_{2,3}$. From the structure of the centralizer of z and the unipotency of U we obtain
$$\delta(v_2) = E_{1,3} + \alpha E_{2,1} + \beta E_{2,3}.$$
It follows $v_1\delta(v_2) = v_3$. Now, we apply $\delta_R(v\delta_R(w)) = \delta_R(v)\delta_R(w)$ to v_1, v_2, which gives
$$\delta(v_3) = \delta(v_1)\delta(v_2) = 0.$$
Direct calculation shows that U is abelian. Hence $d(\mathbf{Z}(U)) = d(U) = 2$. In particular, $(\mu(v_2) - I_4)^2 = 0$ gives $\alpha = \beta = 0$. Finally $(\mu(v_1 + v_2) - I_4)^2 = 0$ gives $\mathrm{char}\,\mathbb{F} = 2$. □

We can identify the direct product $\mathrm{AGL}_{m_1}(\mathbb{F}) \times \mathrm{AGL}_{m_2}(\mathbb{F})$ with the subgroup:

$$\left\{ \begin{pmatrix} 1 & v & w \\ 0 & A & 0 \\ 0 & 0 & B \end{pmatrix} : v \in \mathbb{F}^{m_1},\ w \in \mathbb{F}^{m_2},\ A \in \mathrm{GL}_{m_1}(\mathbb{F}),\ B \in \mathrm{GL}_{m_2}(\mathbb{F}) \right\}.$$

Clearly, in this identification, if U_i are respective regular subgroups of $\mathrm{AGL}_{m_i}(\mathbb{F})$ for $i = 1, 2$ then $U_1 \times U_2$ is a regular subgroup of $\mathrm{AGL}_{m_1+m_2}(\mathbb{F})$.

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For every partition λ of $n+1$ different from (1^{n+1}), we define one or two abelian regular subgroups $S_\lambda, S_\lambda^\sharp$ of $\mathrm{AGL}_n(\mathbb{F})$.

We start with the abelian subgroup:

$$S_{(1+n)} = \mathbf{C}_{\mathrm{AGL}_n(\mathbb{F})}(J_{1+n}).$$

More generally, for a partition λ of $n+1$ such that:

$\lambda = (1+n_1, n_2, \ldots, n_s), \quad s \geq 1, \quad n_i \geq n_{1+i} \geq 1, \quad 1 \leq i \leq s-1,$

we define an abelian regular subgroup S_λ of $\mathrm{AGL}_n(\mathbb{F})$, setting

$$S_\lambda = S_{(1+n_1,\ldots,n_s)} = \prod_{j=1}^{s} S_{(1+n_j)}.$$

In particular, if $\lambda = (2, 1^{n-1})$, then $S_\lambda = \mathcal{T}$.

Lemma

Given a partition $(1+n_1, n_2)$ of $n+1$, with $1+n_1 \geq n_2 > 1$, set:

$$S^\sharp_{(1+n_1,n_2)} = \left\{ \begin{pmatrix} 1 & u & v \\ 0 & \tau_{1+n_1}(u) & w \otimes Du^{\mathrm{T}} \\ 0 & 0 & \tau_{n_2}(v) \end{pmatrix} : u \in \mathbb{F}^{n_1}, v \in \mathbb{F}^{n_2} \right\},$$

where $w = (0, \ldots, 0, 1) \in \mathbb{F}^{n_2}$, $D = \mathrm{antidiag}(1, \ldots, 1) \in \mathrm{GL}_{n_1}(\mathbb{F})$. Then $S^\sharp_{(1+n_1,n_2)}$ is an indecomposable regular subgroup of $\mathrm{AGL}_n(\mathbb{F})$.

Definition

A subgroup H of $\mathrm{AGL}_n(\mathbb{F})$ is indecomposable if there exists no decomposition of $\mathbb{F}^n$ as a direct sum of non-trivial $\pi(H)$-invariant subspaces.

Next, consider a partition μ of $n+1$ such that:

$$\mu = (1+n_1, n_2, \ldots, n_s),\ s \geq 2,\ 1+n_1 \geq n_2 > 1,$$
$$n_i \geq n_{1+i} \geq 1,\ 2 \leq i \leq s-1.$$

We define the abelian regular subgroup $S_\mu^\sharp$ in the following way:

$$S_\mu^\sharp = S^\sharp_{(1+n_1, n_2, n_3 \ldots, n_s)} = S^\sharp_{(1+n_1, n_2)} \times \prod_{j=3}^{s} S_{(1+n_j)}.$$

The regular subgroups $S_\lambda, S_\mu^\sharp$ associated to partitions as above will be called standard regular subgroups. As already mentioned S_λ and $S_\mu^\sharp$ are always abelian.

Theorem

Let $\lambda_1 = (1+n_1, \ldots, n_s), \lambda_2 = (1+m_1, \ldots, m_t)$ be two partitions of $n+1$ as in the first case and let $\mu_1 = (1+a_1, \ldots, a_h), \mu_2 = (1+b_2, \ldots, b_k)$ be two partitions of $n+1$ as in second case. Then

(a) S_{λ_1} *is not conjugate to* $S^{\sharp}_{\mu_1}$;

(b) S_{λ_1} *is conjugate in* $\mathrm{AGL}_{n+1}(\mathbb{F})$ *to* S_{λ_2} *if and only if* $\lambda_1 = \lambda_2$;

(c) $S^{\sharp}_{\mu_1}$ *is conjugate in* $\mathrm{AGL}_{n+1}(\mathbb{F})$ *to* $S^{\sharp}_{\mu_2}$ *if and only if* $\mu_1 = \mu_2$.

THE CLASSIFICATION FOR $\mathrm{AGL}_1(\mathbb{F})$

As shown by Tamburini (2012), the only regular subgroup of $\mathrm{AGL}_1(\mathbb{F})$ is the translation subgroup

$$\mathcal{T} = S_{(2)} = \begin{pmatrix} x_0 & x_1 \\ 0 & x_0 \end{pmatrix}.$$

However, already for $n = 2$, a description becomes much more complicated. Also, restricting to the unipotent case, one may only say that every unipotent regular subgroup is conjugate to:

$$\begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & \sigma(x_1) \\ 0 & 0 & 1 \end{pmatrix}$$

where $\sigma \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{F}, \mathbb{F})$.

A complete set of representatives for the conjugacy classes of regular subgroups U of $\mathrm{AGL}_2(\mathbb{F})$ with linear δ, for any field $\mathbb{F}$:

U	$\mathbb{F}I_{n+1}+U$	$\mathrm{Ker}\,(\Psi)$	
$S_{(3)}$	$\begin{pmatrix} x_0 & x_1 & x_2 \\ 0 & x_0 & x_1 \\ 0 & 0 & x_0 \end{pmatrix}$	$\langle {t_1}^3 \rangle$	indec.
$S_{(2,1)}$	$\begin{pmatrix} x_0 & x_1 & x_2 \\ 0 & x_0 & 0 \\ 0 & 0 & x_0 \end{pmatrix}$	$\langle t_1, t_2 \rangle^2$	

The classification for $\mathrm{AGL}_3(\mathbb{F})$

A complete set of representatives for the conjugacy classes of abelian regular subgroups U of $\mathrm{AGL}_3(\mathbb{F})$, for any field $\mathbb{F}$.

U	$\mathbb{F}I_4 + U$	char $\mathbb{F}$	Ker (Ψ)
$S_{(4)}$	$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ 0 & x_0 & x_1 & x_2 \\ 0 & 0 & x_0 & x_1 \\ 0 & 0 & 0 & x_0 \end{pmatrix}$	any	$\langle {t_1}^4 \rangle$
$S_{(3,1)}$	$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ 0 & x_0 & x_1 & 0 \\ 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & x_0 \end{pmatrix}$	any	$\langle {t_1}^3, {t_2}^2, t_1 t_2 \rangle$
R_λ, $\lambda \in \mathbb{F}^\square$	$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ 0 & x_0 & 0 & x_1 \\ 0 & 0 & x_0 & \lambda x_2 \\ 0 & 0 & 0 & x_0 \end{pmatrix}$	any	$\langle {t_1}^2 - \lambda {t_2}^2, t_1 t_2 \rangle$

U_1^3	$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ 0 & x_0 & 0 & x_2 \\ 0 & 0 & x_0 & x_1 \\ 0 & 0 & 0 & x_0 \end{pmatrix}$	2	$\langle {t_1}^2, {t_2}^2 \rangle$
$S_{(2,1^2)}$	$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ 0 & x_0 & 0 & 0 \\ 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & x_0 \end{pmatrix}$	any	$\langle t_1, t_2, t_3 \rangle^2$

Lemma

U is conjugate to exactly one of the following subgroups:

(a) $N_1 = \begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, if $k(U) = 2$;

(b) $N_2 = \begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & 0 & -x_2 \\ 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, if $k(U) = 1$, $d(U) = 2$ and $\operatorname{char} \mathbb{F} \neq 2$;

(c) $N_{3,\lambda} = \begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & 0 & x_1 + x_2 \\ 0 & 0 & 1 & \lambda x_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $\lambda \in \mathbb{F}^*$, if $k(U) = 1$ and $d(U) = 3$.

A complete set of representatives for the conjugacy classes of indecomposable abelian regular subgroups U of $\mathrm{AGL}_4(\mathbb{F})$, when $\mathbb{F}$ has no quadratic extensions

U	$\mathbb{F}I_5 + U$	char $\mathbb{F}$	Ker (Ψ).
$S_{(5)}$	$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ 0 & x_0 & x_1 & x_2 & x_3 \\ 0 & 0 & x_0 & x_1 & x_2 \\ 0 & 0 & 0 & x_0 & x_1 \\ 0 & 0 & 0 & 0 & x_0 \end{pmatrix}$	any	$\langle {t_1}^5 \rangle$
$S^\sharp_{(3,2)}$	$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ 0 & x_0 & x_1 & 0 & x_2 \\ 0 & 0 & x_0 & 0 & x_1 \\ 0 & 0 & 0 & x_0 & x_3 \\ 0 & 0 & 0 & 0 & x_0 \end{pmatrix}$	any	$\langle {t_1}^3 - {t_2}^2, t_1 t_2 \rangle$

U	$\mathbb{F}I_5 + U$	char $\mathbb{F}$	Ker (Ψ).
U_1^4	$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ 0 & x_0 & x_1 & 0 & x_3 \\ 0 & 0 & x_0 & 0 & 0 \\ 0 & 0 & 0 & x_0 & x_1 \\ 0 & 0 & 0 & 0 & x_0 \end{pmatrix}$	any	$\langle {t_1}^3, {t_1}^2 t_2, {t_2}^2 \rangle$
U_2^4	$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ 0 & x_0 & 0 & 0 & x_1 \\ 0 & 0 & x_0 & 0 & x_3 \\ 0 & 0 & 0 & x_0 & x_2 \\ 0 & 0 & 0 & 0 & x_0 \end{pmatrix}$	any	$\langle {t_1}^2 - t_2 t_3, {t_2}^2, {t_3}^2, t_1 t_2, t_1 t_3 \rangle$

Representatives for the conjugacy classes of decomposable abelian regular subgroups U of $\mathrm{AGL}_4(\mathbb{F})$, when $\mathbb{F}$ has no quadratic extension

U	$\mathbb{F}I_5 + U$	$\mathrm{char}\,\mathbb{F}$	$\mathrm{Ker}\,(\Psi)$
$S_{(4,1)}$	$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ 0 & x_0 & x_1 & x_2 & 0 \\ 0 & 0 & x_0 & x_1 & 0 \\ 0 & 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & 0 & x_0 \end{pmatrix}$	any	$\langle {t_1}^4, {t_2}^2, t_1 t_2 \rangle$
$S_{(3,2)}$	$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ 0 & x_0 & x_1 & 0 & 0 \\ 0 & 0 & x_0 & 0 & 0 \\ 0 & 0 & 0 & x_0 & x_3 \\ 0 & 0 & 0 & 0 & x_0 \end{pmatrix}$	any	$\langle {t_1}^3, {t_2}^3, t_1 t_2 \rangle$

The classification for $AGL_4(\mathbb{F})$

U	$\mathbb{F}I_5 + U$	$\operatorname{char}\mathbb{F}$	$\operatorname{Ker}(\Psi)$
$S_{(3,1,1)}$	$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ 0 & x_0 & x_1 & 0 & 0 \\ 0 & 0 & x_0 & 0 & 0 \\ 0 & 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & 0 & x_0 \end{pmatrix}$	any	$\langle {t_1}^3, {t_2}^2, {t_3}^2, t_1 t_2, t_1 t_3, t_2 t_3 \rangle$
$S^\sharp_{(2,2,1)}$	$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ 0 & x_0 & 0 & x_1 & 0 \\ 0 & 0 & x_0 & x_2 & 0 \\ 0 & 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & 0 & x_0 \end{pmatrix}$	any	$\langle {t_1}^2 - {t_2}^2, {t_3}^2, t_1 t_2, t_1 t_3, t_2 t_3 \rangle$

THE CLASSIFICATION FOR $\mathrm{AGL}_4(\mathbb{F})$

U	$\mathbb{F}I_5 + U$	char $\mathbb{F}$	Ker (Ψ)
$U_1^3 \times S_{(1)}$	$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ 0 & x_0 & 0 & x_2 & 0 \\ 0 & 0 & x_0 & x_1 & 0 \\ 0 & 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & 0 & x_0 \end{pmatrix}$	2	$\langle {t_1}^2, {t_2}^2, {t_3}^2, t_1 t_3, t_2 t_3 \rangle$
$S_{(2,1^3)}$	$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ 0 & x_0 & 0 & 0 & 0 \\ 0 & 0 & x_0 & 0 & 0 \\ 0 & 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & 0 & x_0 \end{pmatrix}$	any	$\langle t_1, t_2, t_3, t_4 \rangle^2$

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