Subgroup Isomorphism Problem and Sylow theorems for units in integral group rings

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Notation and Setting

- G finite group
- $\mathbb{Z}G$ integral group ring over G
- Augmentation map: $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$, $\varepsilon(\sum_{g \in G} z_g g) = \sum_{g \in G} z_g$
- $\mathrm{V}(\mathbb{Z}G)$ group of units of augmentation 1, aka normalized units
- The units of $\mathbb{Z}G$ are $\pm\mathrm{V}(\mathbb{Z}G)$, i.e. it suffices to study $\mathrm{V}(\mathbb{Z}G)$
- For $g \in G$ an element of the form $\pm g$ is called trivial unit.

We want to study the conncetion of finite subgroups of $\mathrm{V}(\mathbb{Z}G)$ and G. In general:

- $\exp(\mathrm{V}(\mathbb{Z}G)) = \exp(G)$ (Cohen-Livingstone '65)
- $U \leq \mathrm{V}(\mathbb{Z}G)$ finite, then $|U|$ divides $|G|$ (Zmud-Kurennoi '67)

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Higman's "plausible" theorem and (SIP)

G. Higman (in his 1940 thesis, § 5)

"The theorems we prove are all partial cases of this plausible theorem: A group of units of finite order in $R(G, C)$ is isomorphic to a group of trivial units."

If G is nilpotent, this is true by a result of A. Weiss ('91).

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There exist groups G and H of order $2^{21} \cdot 97^{28}$ and derived length 4 such that $\mathbb{Z}G \cong \mathbb{Z}H$, but $G \not\cong H$.

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Subgroup Isomorphism Problem (SIP): (W. Kimmerle 2007)

For which finite groups U does the following statement hold: If $\mathrm{V}(\mathbb{Z}G)$ contains a subgroup isomorphic to U, for some finite group G, then G contains a subgroup isomorphic to U.

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Subgroup Isomorphism Problem

Only known counterexamples to (SIP):
Groups based on Hertweck's counterexamples to the Isomorphism Problem.
Only groups known to satisfy (SIP):

- C_{p^n} for p a prime, Cohen-Livingstone '65
- $C_2 \times C_2$, Kimmerle '06 using the Brauer-Suzuki Theorem
- $C_p \times C_p$ for an odd prime p, Hertweck '07, using elementary representation theory
- $C_4 \times C_2$, proof sketched in this talk

These results imply Sylow-like theorems for $\mathrm{V}(\mathbb{Z}G)$:

- If the Sylow p-subgroup of G is cylic or isomorphic to $C_p \times C_p$, then any p-subgroup of $\mathrm{V}(\mathbb{Z}G)$ is isomorphic to a subgroup of G.
- If the Sylow 2-subgroup of G is abelian, (generalized) quaternion or dihedral, then any 2-subgroup of $\mathrm{V}(\mathbb{Z}G)$ is isomorphic to a subgroup of G.

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Strategy

Lemma

If P is a 2-group not containing a subgroup isomorphic to $C_4 \times C_2$, then P is cyclic, elementary-abelian, (generalized) quaternion, dihedral or semidihedral.

Proposition (Dokuchaev-Juriaans '96)

Let $N \trianglelefteq G$ and let U be a group of order comprime to $|N|$. If $\mathrm{V}(\mathbb{Z}G)$ contains a group isomorphic to U, then so does $\mathrm{V}(\mathbb{Z}G/N)$.

$\rightarrow$ To prove (SIP) for $C_4 \times C_2$ one can assume $O_{2'}(G) = 1$ and that the Sylow 2-subgroup of G is dihedral or semidihedral. Such groups were classified by Gorenstein and Walter and by Alperin, Brauer and Gorenstein.

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Methods

For non-solvable G almost no methods are known to study (SIP).
Let $u = \sum_{g \in G} z_g g \in \mathbb{Z}G$ and let x^G be a conjugacy class in G.
Then $\varepsilon_x(u) = \sum_{g \in x^G} z_g$ is called **partial augmentation** of u at x.

For $u \in \mathrm{V}(\mathbb{Z}G)$ of order n we know:

- $\varepsilon_x(u) = 0$, if x is central and $u \neq x$.
- If the order of x does not divide n, then $\varepsilon_x(u) = 0$.
- If $n = p^m$ and x_i are representatives of conj. cl. of elements of order p^k in G, where $k < m$, then $\sum_{x_i} \varepsilon_{x_i}(u) \equiv 0 \mod p$.

Let χ be the extension of an ordinary or p-Brauer character of G to the p-regular elements of $\mathrm{V}(\mathbb{Z}G)$ and let ψ be an ordinary character of a p-regular subgroup U of $\mathrm{V}(\mathbb{Z}G)$. Then:

- $\chi(u) = \sum_{\substack{x^G \\ x\ p-\text{regular}}} \varepsilon_x(u)\chi(x)$.
- $\frac{1}{|U|} \sum_{u \in U} \chi(u)\psi(u^{-1})$ is a non-negative integer.

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Example M_{11}

Let $G = M_{11}$ be the Mathieu group of degree 11. Let $U \leq \mathrm{V}(\mathbb{Z}G)$ such that $U \cong C_4 \times C_2$. Then U contains three involutions and four elements of order 4, say $\alpha^{\pm 1}$ and $\beta^{\pm 1}$.

	$1a$	$2a$	$4a$
χ	10	-2	0

If $u \in U$ is of order 2, then $\chi(u) = \chi(2a)\varepsilon_{2a}(u) = -2$. If $u \in U$ is of order 4, then

$$\chi(u) = \chi(2a)\varepsilon_{2a}(u) + \chi(4a)\varepsilon_{4a}(u) = -2\varepsilon_{2a}(u) \equiv 0 \mod 4.$$

Thus

$$\frac{1}{8}\sum_{u \in U} \chi(u) = \frac{1}{8}\left(10 + 3 \cdot (-2) + 2\chi(\alpha) + 2\chi(\beta)\right)$$

is a non-negative integer, contradicting $2\chi(u) \equiv 0 \mod 8$ for u of order 4.

Does this work for other groups?

Can this method prove (SIP) for other groups?

- For $C_2 \times C_2 \times C_2$ it might, but one needs a reduction to groups being close to being simple.
- It fails for $C_3 \times C_3 \times C_3$ and $G = \mathrm{PSL}(3,3)$ and for the non-abelian group of order 27 and exponent 3 for $G = \mathrm{PSL}(2,3^6)$.
- For cyclic groups not of prime power order it is known to fail in many cases, e.g. for C_6 it fails for $G = \mathrm{PSL}(2,16)$ (this particular case can be solved using another method).

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