Brauer characters of q'- degrees

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Introduction

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Let $\mathrm{Irr}(G)$ be the set of all complex irreducible characters of G and let $\mathrm{IBr}(\mathrm{G})$ be the set of irreducible p-Brauer characters of G.

The celebrated Itô-Michler theorem says that p does not divide $\chi(1)$ for all $\chi \in \mathrm{Irr}(G)$ if and only if G has a normal abelian Sylow p-subgroup.

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Now let q be a prime and assume that q divides the degree of no irreducible p-Brauer character of G.

Indeed, it is known that if $q = p$, then G has a normal Sylow q-subgroup.

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In our first result, we prove that this is true.

Theorem 1.

Let p be a prime and let G be a finite p-solvable group. Let q be a prime and suppose that q divides the degree of no irreducible p-Brauer character of G. Then every p-regular conjugacy class of G meets $N_G(Q)$, where Q is a Sylow q-subgroup of G.

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We write $\Delta_H(G)$ for the set of all H-derangements of G.

If H is core-free in G, then G is a permutation group acting on the right coset space $\Omega = G/H$ with point stabilizer H and $\Delta(G) = \Delta_H(G)$ is the set of all derangements or fixed-point-free elements of G on Ω.

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$$\Delta_H(G) = G \setminus \cup_{g \in G} H^g.$$

With this concept, Theorem 1 can be restated as follows:

Let p and q be primes and let G be a finite p-solvable group and Q a Sylow q-subgroup of G. If q divides the degree of no irreducible p-Brauer character of G, then all $N_G(Q)$-derangements of G have order divisible by p.

Using the Itô-Michler Theorem for Brauer characters and a result of Fein, Kantor, and Schacher which states that every finite permutation group of degree > 1 contains a derangement of prime power order, it is easy to see that for a finite group G, every p-Brauer character of G has p'-degree if and only if every $N_G(P)$-derangement of G, for some Sylow p-subgroup P of G, has order divisible by p.

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Unfortunately, this does not hold true when q is different from p.

There are examples that show that the p-solvable assumption on G in Theorem 1 is necessary.

Notice that the condition that every $N_G(P)$-derangement of G, for some Sylow p-subgroup P of G, has order divisible by p is enough to characterize the groups G where every p-Brauer character has p'-degree.

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On the other hand, every $\{p, q\}$-group G has the property that every p-regular conjugacy of G meets $N_G(Q)$ where Q is a Sylow q-subgroup of G. (In this case, every p-regular element of G has q-power order and so, every p-regular conjugacy class of G meets Q.)

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Since there exist $\{p, q\}$-groups with irreducible p-Brauer characters that do not have q'-degrees, the condition that every p-regular class of G meets $N_G(Q)$ for some Sylow q-subgroup of G is not sufficient to characterize the p-solvable groups with q dividing the degree of no irreducible p-Brauer character.

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Our next goal is to obtain just such a characterization.

Manz and Wolf have previously studied these groups.

Manz and Wolf proved that if G is a p-group so that all irreducible p-Brauer characters have degrees not divisible by q, then $\mathbf{O}^{q'}(G)$ is solvable.

If we make the assumption that a Sylow q-subgroup of G is abelian, then we are able to characterize the groups with all irreducible p-Brauer characters having q'-degree by adding the condition that $\mathbf{O}^{q'}(G)$ is solvable.

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Theorem 2.

Let p and q be distinct primes and suppose that G is p-solvable and a Sylow q-subgroup Q of G is abelian. Then $q \nmid \varphi(1)$ for all $\varphi \in \mathrm{IBr}(\mathrm{G})$ if and only if the following conditions hold:

(1) $x^G \cap N_G(Q) \neq \emptyset$ *for all p-regular elements $x \in G$;*

(2) $\mathbf{O}^{q'}(G)$ *is solvable;*

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1. in a q-series for G, the q-factors are abelian,
2. the q-length of $G/\mathbf{O}_{p,q}(G)$ is at most 1,
3. and the Sylow q-subgroups of G are abelian or metabelian.

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There exist examples of groups that meet the conclusion of Theorem 1 and the conditions of Manz and Wolf, yet have irreducible p-Brauer characters whose degrees are divisible by q.

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There exist examples of groups that meet the conclusion of Theorem 1 and the conditions of Manz and Wolf, yet have irreducible p-Brauer characters whose degrees are divisible by q.

Thus, to obtain a characterization, we need a further condition beyond the one stated in Theorem 1 and the conditions found by Manz and Wolf.

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We prove that $C_G(M/N)$ is a subgroup of G and it contains M whenever M/N is abelian.

We will also prove that if $M \leq K \leq G$ and $K' \leq N$, then $K \leq C_G(M/N)$.

The characterization we obtain is:

Theorem 3.

Let p and q be distinct primes and suppose that G is p-solvable with $\mathbf{O}_p(G) = 1$. Let $L := \mathbf{O}^{q'}(G)$ and let $Q \leq L$ be a Sylow q-subgroup of G. Then $q \nmid \varphi(1)$ for all $\varphi \in \mathrm{IBr(G)}$ if and only if the following conditions hold:

(1) $x^G \cap N_G(Q) \neq \emptyset$ *for all p-regular elements $x \in G$;*

(2) *L is solvable;*

(3) $\mathbf{O}_q(L)$ *is abelian;*

(4) *For every normal subgroup N of $\mathbf{O}_q(L)$ with $\mathbf{O}_q(L)/N$ cyclic, the following hold:*

   1. *there exists an element $g \in L$ such that $(Q^g)' \leq N$*
   2. *Every p-regular conjugacy class of $C/\mathbf{O}_q(L)$ meets $N_{C/\mathbf{O}_q(L)}(Q^g/\mathbf{O}_q(L))$, where $C = C_L(\mathbf{O}_q(L)/N)$.*

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To apply Theorem 2, we need the condition that every p-regular class intersect the normalizer of a Sylow q-subgroup.

Idea of Proof of Theorem 1

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In particular, all p-Brauer characters of $\mathbf{O}^{q'}(G)$ have q'-degree.

We show the converse of this holds when G/N is p-solvable.

Lemma 1.

Let p and q be distinct primes and suppose that $G/\mathbf{O}^{q'}(G)$ is p-solvable. Then $q \nmid \varphi(1)$ for all $\varphi \in \mathrm{IBr}(G)$ if and only if $q \nmid \beta(1)$ for all $\beta \in \mathrm{IBr}(\mathbf{O}^{q'}(G))$.

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Proof: By the discussion above, it suffices to show that if all irreducible p-Brauer characters of $L := \mathbf{O}^{q'}(G)$ have q'-degree, then $q \nmid \varphi(1)$ for all $\varphi \in \mathrm{IBr(G)}$.

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Let $\varphi \in \mathrm{IBr(G)}$ and let $\theta \in \mathrm{IBr(L)}$ be an irreducible constituent of φ_L.

By a Theorem of Dade (this requires p-solvability), we have $\varphi(1)/\theta(1)$ divides $|G/L|$.

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Since G/L is a q'-group and $q \nmid \theta(1)$ by our assumption, we deduce that $q \nmid \varphi(1)$.

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Since G/L is a q'-group and $q \nmid \theta(1)$ by our assumption, we deduce that $q \nmid \varphi(1)$.

This proves the lemma.

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Lemma 2.

Let H be a proper subgroup of a finite group G and L be a normal subgroup of G such that $G = HL$. If T is a proper subgroup of L containing $H \cap L$, then $\Delta_T(L) \subseteq \Delta_H(G)$.

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(1) *If $G = HL$ and (G, H) satisfies $\mathcal{D}_p$ then so does $(L, H \cap L)$.*

(2) *If L is a p-group or p'-group, (G, H) satisfies $\mathcal{D}_p$ and $G \neq HL$, then $(G/L, HL/L)$ also satisfies $\mathcal{D}_p$.*

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(2) *If L is a p-group or p'-group, (G, H) satisfies $\mathcal{D}_p$ and $G \neq HL$, then $(G/L, HL/L)$ also satisfies $\mathcal{D}_p$.*

(3) *If $H \leq K < G$ and (G, H) satisfies $\mathcal{D}_p$, then (G, K) satisfies $\mathcal{D}_p$.*

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(3) *If $H \leq K < G$ and (G, H) satisfies $\mathcal{D}_p$, then (G, K) satisfies $\mathcal{D}_p$.*
(4) *If $L \trianglelefteq G$ such that $L \leq H$ and $(G/L, H/L)$ satisfies $\mathcal{D}_p$ then (G, H) satisfies $\mathcal{D}_p$.*

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Lemma 4.

Let p and q be distinct primes. Let Q be a Sylow q-subgroup of G and let $L \trianglelefteq G$. Suppose that $x^G \cap N_G(Q) \neq \emptyset$ for all p-regular elements x of G. Then $x^L \cap N_L(Q \cap L) \neq \emptyset$ for all p-regular elements x of L. In particular, if $Q \leq L$, then $x^L \cap N_L(Q) \neq \emptyset$ for all p-regular elements x of L.

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So, we may assume that $N_L(U)$ is a proper subgroup of L which implies that both H and $N_G(U)$ are proper subgroups of G.

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Clearly, (G, H) satisfies $\mathcal{D}_p$ by the hypothesis, so $(G, N_G(U))$ satisfies $\mathcal{D}_p$ by Lemma 3(3).

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Clearly, (G, H) satisfies $\mathcal{D}_p$ by the hypothesis, so $(G, N_G(U))$ satisfies $\mathcal{D}_p$ by Lemma 3(3).

Now part (1) of Lemma 3 implies that $(L, L \cap N_G(U))$ satisfies $\mathcal{D}_p$ or $(L, N_L(U))$ satisfies $\mathcal{D}_p$ as wanted.

We first prove Theorem 1 under the additional hypothesis that $G = Q\mathbf{O}_{q'}(G)$ where Q is a Sylow q-subgroup of G.

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Lemma 5.

Let p and q be distinct primes and let $Q \in \mathrm{Syl}_q(G)$. Suppose that $G = Q\mathbf{O}_{q'}(G)$ and that $q \nmid \varphi(1)$ for all $\varphi \in \mathrm{IBr(G)}$. Let $K = \mathbf{O}_{q'}(G)$ and $H = N_G(Q)$. Then

1. *Q is abelian and $x^K \cap C_K(Q) \neq \emptyset$ for all p-regular elements $x \in K$;*
2. *$x^G \cap N_G(Q)$ is non-empty for all p-regular elements $x \in G$.*

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We also prove that $\mathbf{O}_p(G) = 1$.

Let $L = \mathbf{O}^{q'}(G)$. Prove that $L = QK$ where $K = \mathbf{O}_{q'}(L)$ is solvable.

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Now fix a p-regular element $y \in HN$.

Then $y = hn$ for some $h \in H$ and $n \in N$.

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Hence $Q^{k^{-1}} \leq C_{QK}(n)$, and thus by Sylow theorem, $Q^{k^{-1}} = Q^l$ for some $l \in C_{QK}(n)$ so $Q^{lk} = Q$ and hence $lk \in H$.

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Since $n^{lk} = n^k$ and $h^{lk} \in H$, we obtain that

$$y^{lk} = (hn)^{lk} = h^{lk} n^{lk} = h^{lk} n^k \in H.$$

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$$y^{lk} = (hn)^{lk} = h^{lk} n^{lk} = h^{lk} n^k \in H.$$

Therefore, we have shown that $y^G \cap H$ is not empty.

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Let $G = \mathrm{SL}_2(2^f)$.

Then $N_G(Q) \cong \mathrm{D}_{2(2^f+1)}$ and all irreducible 2-Brauer characters of G have 2-power degree.

In particular, $q \nmid \varphi(1)$ for all 2-Brauer characters $\varphi \in \mathrm{IBr(G)}$, but $N_G(Q)$ contains no element of order $2^f - 1$.

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Then $2 \nmid \varphi(1)$ for all $\varphi \in \mathrm{IBr(G)}$ but $N_G(Q) = Q$ contains no odd p-regular element of G.

Finally, we present examples of groups that satisfy the conclusion of Theorem 1 and the conditions of Manz and Wolf, yet have irreducible p-Brauer characters whose degrees are divisible by q.

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Also, by Burnside's theorem, we know that any $\{p, q\}$-group is necessarily solvable.

Thus, it suffices to find a $\{p, q\}$-group G where in the q-series for G, the q-factors are abelian, the q-length of $G/\mathbf{O}_{p,q}(G)$ is at most 1, and the Sylow q-subgroups are metabelian, and there exists a p-Brauer character whose degree is divisible by q.

A specific example when $p = 3$ and $q = 2$ can be found by taking the semidirect product of S_3 acting on two copies of the Klein 4-group where the action is found in S_4.

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We claim that for any two distinct primes p and q, the iterated wreath product of Z_q by Z_p and then Z_q again will yield an example, but we leave the details as an exercise.