Groups equal to a product of three conjugate subgroups

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The question

G - a finite group

A *factorization of* G *by conjugates of* $A < G$ is any representation of G as a setwise product

$$G = A^{g_1} \cdots A^{g_m}$$

where $g_i \in G$ for all $1 \leq i \leq m$.

Q: What can be said about the minimal m when A varies over all proper subgroups of G?

Answers

- Conjugate factorizations of G exist if and only if G is non-nilpotent.

Henceforth assume G is non-nilpotent.

- $m \geq 3$
- G solvable: The minimal m depends on G and is unbounded.

Example: $G = D_{2p}$, $p \geq 3$ a prime. Can only use $A = \langle r \rangle$ where r is a reflection. The minimal m is

$$\lceil \log_2 p \rceil + 1.$$

The non-solvable answer

Thm: If G is a finite non-solvable group then there exists $A < G$ such that G is a product of three conjugates of A.

What goes into the proof

1. Connection between products of conjugates of A and products of double cosets of A.

In particular:

$$G = (AxA)^2 \iff G = AA^xA^{x^2}$$

Thm: If G is a finite non-solvable group then there exists $A < G$ such that $G = (AxA)^2$.

Proof ingredients continued

2. Reduction to almost simple groups.
3. Use the classification of the finite simple groups.
4. Alternating groups: $G = Alt(n), A = Alt(n-1), n \geq 5$.

G acts 2-transitively on $\{1, \ldots, n\}$
with point stabilizer A

$$\Longleftrightarrow$$

$$G = A \cup AxA, x \notin A$$

$$\Longrightarrow \quad G = (AxA)^2$$

Groups with a BN- pair

5. Groups of Lie type: G is a simple group of Lie type and $A = B$ is a Borel subgroup of G.

In particular, G is a group with a BN- pair and a finite Weyl group W.

Thm: Let G be a group with a BN- pair and a finite Weyl group W. Let $w_0 \in W$ be the unique longest element of W, and $n_0 \in N$ be such that $n_0 H = w_0$. Then

$$G = (B n_0 B)^2 = B B^{n_0} B.$$

Hecke algebra of the double cosets

6. Sporadic groups: Use Hecke algebra of the double cosets. G be a finite group, $A \leq G$.

$\{x_1, \dots, x_r\} \subseteq \mathrm{G}$ a complete set of distinct A double coset representatives. "Represent" Ax_jA by:

$$e_j := \frac{1}{|A|} \sum_{y \in Ax_jA} y \in \mathbb{Q}[G], \qquad 1 \leq j \leq r.$$

$\{e_1, \dots, e_r\}$ is a basis of a subring of $\mathbb{Q}[G]$:

$$e_i e_j = \sum_{k=1}^{r} c_{ijk}\, e_k,$$

The c_{ijk} are non-negative integers.

Hecke algebra continued

$$G = \left(Ax_jA\right)^2 \Leftrightarrow c_{jjk} > 0, \forall 1 \leq k \leq r.$$

The c_{ijk} can be computed efficiently using the complex representation theory of G, if the permutation character 1_A^G is multiplicity free.

GAP package “mfer” (T. Breuer, I. Höhler and J. Müller) has all necessary data to carry these computations for the sporadic groups.

Except...

Odd man out

7. $G = O'N$. Here there is no multiplicity free permutation character available that also takes care of $Aut(G)$. Choosing $A = J_1$ we implemented a probabilistic algorithm in MAGMA that verifies the existence of $x \in G$ such that $G = (AxA)^2$.

<u>Open Question</u>: Can one classify solvable groups which are equal to a product of three conjugates of a proper subgroup?

Papers

1. M. Garonzi, D. Levy, “Factorizing a Finite Group into Conjugates of a Subgroup”, Journal of Algebra, vol 418 (2014), 129-141.
2. J. Cannon, M. Garonzi, D. Levy, A. Maróti, I. I. Simion, “Groups equal to a product of three conjugate subgroups”. To appear in Israel Journal of Mathematics.