Homogenous Monomial Groups

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29 March-3 April 2016

Dedicated to the memories of

Guido Zappa, Mario Curzio
and
Wolfgang Kappe

Representations of groups

There are three principal types of representations of groups, each with its particular field of usefulness, are the following:

(1) Permutation representation of groups.
(2) Monomial representation of groups.
(3) Linear or matrix representation of groups.

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Oystein Ore

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These three types of representations correspond to an embedding of the group in the following groups:

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The symmetric group and the full linear group have been exhaustively investigated and many of their principal properties are known.

A similar study does not seem to exist for complete monomial group.

Monomial groups

O. Ore; *Theory of monomial groups*, Trans. Amer. Math. Soc. **51**, (1942) 15–64.

Monomial groups

Let us recall the definition of a monomial group.

Monomial groups

Let H ba an arbitrary group and let $n \in \mathbb{N}$. A monomial permutation over H is a linear transformation

$$\gamma = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ h_1 x_{i_1} & h_2 x_{i_2} & \dots & h_n x_{i_n} \end{pmatrix}$$

where each variable is changed into some other variable multiplied by an element of H.

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where each variable is changed into some other variable multiplied by an element of H.

The elements $h_i \in H$ will be called **factors or multipliers** in γ.
The multiplication $h_i x_{i_j}$ is a formal multiplication satisfying $(h_i h_j) x_k = h_i (h_j x_k)$.

Monomial groups

If η is another monomial permutation

$$\eta = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ a_1 x_{j_1} & a_2 x_{j_2} & \dots & a_n x_{j_n} \end{pmatrix}$$

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$$= \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ h_1 a_{i_1} x_{j_{i_1}} & h_2 a_{i_2} x_{j_{i_2}} & \dots & h_n a_{i_n} x_{j_{i_n}} \end{pmatrix}$$

Monomial groups

For

$$\gamma = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ h_1 x_{i_1} & h_2 x_{i_2} & \dots & h_n x_{i_n} \end{pmatrix}$$

$$\gamma^{-1} = \begin{pmatrix} x_{i_1} & x_{i_2} & \dots & x_{i_n} \\ h_1^{-1} x_1 & h_2^{-1} x_2 & \dots & h_n^{-1} x_n \end{pmatrix}$$

Monomial groups

The set $\Sigma_n(H)$ of all monomial permutations over H of variables $x_1, \ldots, x_n$ with multiplication defined as above forms a group.

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The set $\Sigma_n(H)$ of all monomial permutations over H of variables $x_1, \ldots, x_n$ with multiplication defined as above forms a group.

This group $\Sigma_n(H)$ is called **complete monomial group of degree** n.

The set of all monomial permutations of the form

$$\theta = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ x_{i_1} & x_{i_2} & \dots & x_{i_n} \end{pmatrix} = \begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix}$$

forms a subgroup $\Sigma_n(\{1\})$ of $\Sigma_n(H)$ and it is isomorphic to the symmetric group on n letters.

Monomial groups

The monomial permutations of the form

$$\eta = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ h_1 x_1 & h_2 x_2 & \dots & h_n x_n \end{pmatrix} = [h_1, h_2, \dots, h_n]$$

where $h_i \in H$ forms a subgroup of $\Sigma_n(H)$ which is isomorphic to the direct product $H \times \ldots \times H$ of n copies of H.

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where $h_i \in H$ forms a subgroup of $\Sigma_n(H)$ which is isomorphic to the direct product $H \times \dots \times H$ of n copies of H.

Every monomial permutation can be written uniquely as a product of monomial permutation and a product (multiplication).

Monomial groups

For example the element γ can be written as

$$\begin{pmatrix} x_1 & x_2 & \dots & x_n \\ h_1 x_{i_1} & h_2 x_{i_2} & \dots & h_n x_{i_n} \end{pmatrix} = \begin{pmatrix} x_1 & \dots & x_n \\ x_{i_1} & \dots & x_{i_n} \end{pmatrix} \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ h_1 x_1 & h_2 x_2 & \dots & h_n x_n \end{pmatrix}$$

$$= \begin{pmatrix} x_1 & \dots & x_n \\ x_{i_1} & \dots & x_{i_n} \end{pmatrix} [h_1, \dots, h_n]$$

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$$= \begin{pmatrix} x_1 & \dots & x_n \\ x_{i_1} & \dots & x_{i_n} \end{pmatrix} [h_1, \dots, h_n]$$

Moreover one may observe that, when we take the conjugate of a multiplication $[h_1, \dots, h_n]$ by a permutation $\sigma \in \Sigma_n(\{1\})$, we have that the coordinates of the multiplication is permuted. Indeed $\sigma^{-1}[h_1, \dots, h_n]\sigma = [h_1, \dots, h_n]^\sigma = [h_{1^\sigma}, \dots, h_{n^\sigma}]$ hence we have

$$\Sigma_n(H) \cong H \wr S_n \cong (H \times \dots H) \rtimes S_n$$

The wreath product is the permutational wreath product.

Monomial groups

Let G be a group which has a subgroup H of index n in G.

Each element g in G permutes the cosets and there are factors coming from H.
Namely

Monomial groups

$$Hx_i.g = Hx_ig$$

where

$$\{Hx_i|\ i = 1, \ldots, n\}$$

is the set of right cosets of H in G and

$$\{x_i|\ i = 1, \ldots, n\}$$

is the set of right coset representatives of H in G.

Then each $g \in G$ determines a permutation $\pi(g)$ of the right cosets and $x_i.g = h_i(g)x_{i\pi(g)}$ and n elements $h_i(g)$ in H.

Monomial groups

Then the map

$$G \to GL(n, \mathbb{Z}H)$$

$$g \to Diag(h_1(g), h_2(g), \ldots, h_n(g))\pi(g)$$

Monomial groups

$$\begin{pmatrix} h_1(g) & & & \\ & h_2(g) & & \\ & & \ddots & \\ & & & h_n(g) \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & & & 1 \\ 1 & & \ddots & & \\ 0 & 0 & & 1 & 0 \\ 0 & 0 & & & 0 \end{pmatrix}}$$

defines a monomorphism from G into $GL(n, \mathbb{Z}H)$ where $\mathbb{Z}H$ is the group ring of H over the ring of integers.

Monomial groups

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defines a monomorphism from G into $GL(n, \mathbb{Z}H)$ where $\mathbb{Z}H$ is the group ring of H over the ring of integers.

$\Sigma_n(H)$ is isomorphic to the subgroup of monomial matrices in $GL(n, \mathbb{Z}(H))$.

Monomial groups

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Example. Indeed centralizer of an element say $(12)(34)(56) \in S_6$ is

$$C_{S_6}((12)(34)(56)) \cong ((Z_2 \wr S_3) \cong \Sigma_3(Z_2)$$

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Therefore as Ore suggested monomial groups appear naturally as centralizers of elements in symmetric groups.

Monomial groups

A monomial permutation of the form

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where $a_i \in H$ is called a **cycle**.

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η is written in the cycle form $(a_1x_2, a_2x_3, \dots, a_kx_1)$

Monomial groups

As in the symmetric groups one can write each monomial permutation as a product of commuting disjoint cycles.

Example Let $n = 5$ and $H = S_3$.

$$\gamma = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ (123)x_3 & (234)x_2 & (34)x_1 & (134)x_5 & (23)x_4 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 & x_3 \\ (123)x_3 & (34)x_1 \end{pmatrix} \begin{pmatrix} x_2 \\ (234)x_2 \end{pmatrix} \begin{pmatrix} x_4 & x_5 \\ (134)x_5 & (23)x_4 \end{pmatrix}$$

$$= \big(\ (123)x_3 \ \ (34)x_1) \quad ((134)x_5 \ \ (23)x_4) \ \ ((234)x_2)\ \big)$$

Monomial groups

As we mentioned before each monomial can be written as a product of disjoint cycles. Now for each cycle

$$\gamma = \begin{pmatrix} x_1 & x_2 & \dots & x_m \\ c_1x_2 & c_2x_3 & \dots & c_mx_1 \end{pmatrix} = (c_1x_2, c_2x_3, \dots, c_mx_1)$$

where $c_i \in H$, the m^{th} power of γ is

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where $c_i \in H$, the m^{th} power of γ is

$$\gamma^m = \begin{pmatrix} x_1 & x_2 & \dots & x_m \\ \Delta_1 x_1 & \Delta_2 x_2 & \dots & \Delta_m x_n \end{pmatrix}$$

where

$$\Delta_1 = c_1c_2\dots c_m, \quad \Delta_2 = c_2c_3\dots c_mc_1, \ \dots,$$

$$\Delta_m = c_mc_1\dots c_{m-1}$$

Monomial groups

The elements $\Delta_i \in H$ are called the **determinants** of γ. As you observed, the determinants are all conjugate as

$$\Delta_2 = c_1^{-1}\Delta_1 c_1, \ldots \Delta_m = c_{m-1}^{-1}\Delta_{m-1}c_{m-1}, \Delta_1 = c_m^{-1}\Delta_m c_m$$

So one observes that for each cycle there is a unique determinant class in H.

Diagonal Embedding of Monomial groups

Let Π be the set of sequences consisting of prime numbers. Let $\xi \in \Pi$ and $\xi = (p_1, p_2, \ldots)$ be a sequence consisting of not necessarily distinct primes p_i.

From the given sequence ξ we may obtain a divisible sequence $(n_1, n_2, \ldots n_i, \ldots)$ where

$$n_1 = p_1, \text{ and } \; n_{i+1} = p_{i+1} n_i$$

we have

$$n_i | n_{i+1}$$

for all $i = 1, 2, \ldots$.

Diagonal Embedding of Monomial groups

We may embed a complete monomial group $\Sigma_{n_i}(H)$ diagonally into $\Sigma_{n_{i+1}}(H)$ as follows.

$$d^{p_{i+1}} : \Sigma_{n_i}(H) \to \Sigma_{n_{i+1}}(H)$$

Diagonal Embedding of Monomial groups

Given $\gamma \in \Sigma_{n_i}(H)$ where $\gamma = \begin{pmatrix} x_1 & x_2 & \dots & x_{n_i} \\ h_1 x_{j_1} & h_2 x_{j_2} & \dots & h_{n_i} x_{j_{n_i}} \end{pmatrix}$
we define $d^{p_{i+1}}(\gamma) =$

$$\left(\begin{array}{cccc|cccc|c} x_1 & x_2 & \dots & x_{n_i} & x_{n_i+1} & x_{n_i+2} & \dots & x_{2n_i} & \dots \\ h_1 x_{j_1} & h_2 x_{j_2} & \dots & h_{n_i} x_{j_{n_i}} & h_1 x_{n_i+j_1} & h_2 x_{n_i+j_2} & \dots & h_{n_i} x_{n_i+j_{n_i}} & \dots \end{array} \right)$$

$$\begin{pmatrix} \dots & x_{mn_i+k} & \dots \\ \dots & h_k x_{mn_i+j_k} & \dots \end{pmatrix}$$

where $n_{i+1} = n_i p_{i+1}$ and $\xi = (p_1, p_2, \dots p_i \dots)$ is a sequence of not necessarily distinct primes. This embedding

corresponds to strictly diagonal embedding of $\Sigma_{n_i}(H)$ into $\Sigma_{n_{i+1}}(H)$.

According to the given sequence of primes we continue to embed

$$d^{p_{i+1}} : \Sigma_{n_i}(H) \to \Sigma_{n_{i+1}}(H)$$

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$$\{1\} \stackrel{d^{p_1}}{\to} \Sigma_{n_1}(H) \stackrel{d^{p_2}}{\to} \Sigma_{n_2}(H) \stackrel{d^{p_3}}{\to} \Sigma_{n_3}(H) \stackrel{d^{p_4}}{\to} \dots$$

$$\{1\} \stackrel{d^{p_1}}{\to} A_{n_1}(H) \stackrel{d^{p_2}}{\to} A_{n_2}(H) \stackrel{d^{p_3}}{\to} A_{n_3}(H) \stackrel{d^{p_4}}{\to} \dots$$

where $n_i = n_{i-1}p_i, \quad i = 1, 2, 3 \dots$ and $\Sigma_{n_i}(H)$ is the complete monomial group on n_i letters over the group H and $A_{n_i}(H)$ is the monomial alternating group on n_i letters over H and $n_0 = 1$.

Monomial groups

The direct limit groups obtained from the above construction are called **homogenous monomial group** over the group H and denoted by $\Sigma_{\xi}(H)$ and homogenous alternating group $A_{\xi}(H)$ over H respectively.

$$\Sigma_{\xi}(H) \cong \bigcup_{i=1}^{\infty} \Sigma_{n_i}(H) \cong S(\xi) \ltimes B$$

where B is the base group which is isomorphic to direct product of the group H.

Centralizers of elements in $\Sigma_\xi(H)$

Question 1. Find necessary and sufficient condition for two elements to be conjugate in $\Sigma_\xi(H)$.

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Question 3. When two groups $\Sigma_{\xi_1}(H)$ and $\Sigma_{\xi_2}(H)$ are isomorphic.

Centralizers of elements in $\Sigma_\xi(H)$

For Question 1, by taking the conjugate of a cycle

$$\gamma = \begin{pmatrix} x_1 & x_{i_1} & \dots & x_{i_m} \\ c_1 x_{i_1} & c_2 x_{i_2} & \dots & c_m x_1 \end{pmatrix}$$

by an element of $\Sigma_n(H)$ we may write it in the form

$$\delta = \begin{pmatrix} x_1 & x_{i_1} & \dots & x_{i_m} \\ x_{i_1} & x_{i_2} & \dots & a x_1 \end{pmatrix}$$

where $a \in H$ is an element in the determinant class of γ.

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where $a \in H$ is an element in the determinant class of γ.

δ is called a **normal form** of γ.

Any monomial permutation ρ is similar to a product of cycles without common variables, so $\rho = \gamma_1 \dots \gamma_r$ where each cycle is in normal form.

Conjugation of elements in $\Sigma_\xi(H)$

Lemma 1 (Ore)

The necessary and sufficient condition for two monomial cycles to be conjugate in $\Sigma_n(H)$ is that they shall have the same length and the same determinant class.

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The necessary and sufficient condition for two monomial cycles to be conjugate in $\Sigma_n(H)$ is that they shall have the same length and the same determinant class.

Therefore two monomial permutations are conjugate if and only if the cycles in their cycle decomposition may be made correspond in such a manner that corresponding cycles have he same length and determinant class.

Conjugation of elements

The direct limit group can be written as

$$\Sigma_\xi(H) = \bigcup_{i=1}^{\infty} \Sigma_{n_i}(H).$$

For an element $\gamma \in \Sigma_\xi(H)$, we define the short cycle type of γ as the cycle type of γ in the smallest n_i where $\gamma \in \Sigma_{n_i}(H)$.
For the short cycle type, we put an order according to the length and the determinant class.

Conjugation of elements

By type of a monomial permutation γ we have two variables length of cycle and determinant class.

$$t(\gamma) = (a_{11}r_1, a_{12}r_1, \dots a_{1i_1}r_1, a_{21}r_2, a_{22}r_2, \dots, a_{2i_2}r_2 \dots, a_{li_l}r_l)$$

a_{ij} is the representative of conjugacy class in H and r_i is the number of cycles of length i in the cycle decomposition of γ with determinant class a_{ij}.

Conjugation of elements

Therefore by using the above lemma, we may state the following:

Lemma 2

Two elements of $\Sigma_{\xi}(H)$ are conjugate in $\Sigma_{\xi}(H)$ if and only if they have the same cycle type in $\Sigma_{n_i}(H)$ for some n_i dividing ξ.

Steinitz numbers

For the centralizers of elements and isomorphism question, we now recall Steinitz numbers (supernatural numbers).

Steinitz numbers

Recall that the formal product $n = 2^{r_2}3^{r_3}5^{r_5}\ldots$ of prime powers with $0 \leq r_k \leq \infty$ for all k is called a **Steinitz number** (supernatural number).

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The set of Steinitz numbers form a partially ordered set with respect to division, namely if $\alpha = 2^{r_2}3^{r_3}5^{r_5}\dots$ and $\beta = 2^{s_2}3^{s_3}5^{s_5}\dots$ be two Steinitz numbers, then $\alpha|\beta$ if and only if $r_p \leq s_p$ for all prime p.

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Moreover they form a lattice if we define meet and join as
$\alpha \wedge \beta = 2^{min\{r_2,s_2\}}3^{min\{r_3,s_3\}}5^{min\{r_5,s_5\}}\dots$ and
$\alpha \vee \beta = 2^{max\{r_2,s_2\}}3^{max\{r_3,s_3\}}5^{max\{r_5,s_5\}}\dots$.

Steinitz numbers

For each sequence ξ, we define a Steinitz number

$$Char(\xi) = 2^{r_2} 3^{r_3} \dots p_i^{r_{p_i}} \dots$$

where r_{p_i} is the number of times that the prime p_i repeat in ξ. If it repeats infinitely often, then we write p_i^{∞}.

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For a group $\Sigma_\xi(H)$ obtained from the sequence ξ we define $Char(\Sigma_\xi(H)) = Char(\xi)$.

Steinitz numbers

For each Steinitz number ξ we can define a homogenous monomial group $\Sigma_\xi(H)$ and for each homogenous monomial group $\Sigma_\xi(H)$ we have a Steinitz number.

Centralizers

Since every cycle is conjugate to a cycle in the normal form and centralizers of conjugate elements are conjugate (isomorphic), we may assume that we have the cycles in the normal form.

Centralizers

Since every cycle is conjugate to a cycle in the normal form and centralizers of conjugate elements are conjugate (isomorphic), we may assume that we have the cycles in the normal form.

Moreover as disjoint cycles commute, we find centralizer of a cycle of determinant class $a \in H$ repeated m times, then centralizer of an arbitrary cycle will be direct product of the centralizers for each distinct cycle.

Centralizers

First we have only one cycle.

Let $\gamma = \begin{pmatrix} x_1 & \dots & x_m \\ x_2 & \dots & ax_1 \end{pmatrix}$ be a cycle in the normal form and $a \in H$.
Then the centralizer of γ in $\Sigma_m(H)$ is isomorphic to

$$C_a = C_H(a)\langle\gamma\rangle$$

Centralizers

Assume that the cycle is repeated s times with the same determinant class $a \in H$. Then
$(\gamma) =$

$$\underbrace{\begin{pmatrix} x_1 & x_2 & \dots & x_m \\ x_2 & x_3 & \dots & ax_1 \end{pmatrix}\begin{pmatrix} x_{m+1} & \dots & x_{2m} \\ x_{m+2} & \dots & ax_{m+1} \end{pmatrix}\dots\begin{pmatrix} x_{(s-1)m+1} & \dots & x_{sm} \\ x_{(s-1)m+2} & \dots & ax_{(s-1)+1} \end{pmatrix}}_{s-times}$$

Then

$$C_{\Sigma_{sm}(H)}(\gamma) \cong (C_H(a)\langle\gamma\rangle) \wr S_s \cong \Sigma_s(C_H(a)\langle\gamma\rangle)$$

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Assume that the cycle is repeated s times with the same determinant class $a \in H$. Then
$(\gamma) =$

$$\underbrace{\begin{pmatrix} x_1 & x_2 & \dots & x_m \\ x_2 & x_3 & \dots & ax_1 \end{pmatrix} \begin{pmatrix} x_{m+1} & \dots & x_{2m} \\ x_{m+2} & \dots & ax_{m+1} \end{pmatrix} \dots \begin{pmatrix} x_{(s-1)m+1} & \dots & x_{sm} \\ x_{(s-1)m+2} & \dots & ax_{(s-1)+1} \end{pmatrix}}_{s-times}$$

Then

$$C_{\Sigma_{sm}(H)}(\gamma) \cong (C_H(a)\langle\gamma\rangle) \wr S_s \cong \Sigma_s(C_H(a)\langle\gamma\rangle)$$

Then by using this we may find the structure of centralizer of an arbitrary element in $\Sigma_\xi(H)$.

Centralizer Special case

Theorem 3

Let γ be an element of $\Sigma_\xi(H)$ where γ is a product of cycles with the same determinant class $a \in H$ of length m repeated s times and principal beginning of γ is contained in $\Sigma_{n_i}(H)$ where $n_i = ms$. Then

$$C_{\Sigma_\xi(H)}(\gamma) \cong C_a(\Sigma_{\xi_1}(C_a))$$

where $\xi_1 = \frac{Char(\xi)}{n_i} s$.

Theorem 4

Let ρ be an element of $\Sigma_\xi(H)$ with principal beginning is in $\Sigma_{n_k}(H)$ with its normal form $\rho = \lambda_1 \dots \lambda_l$, where $\lambda_i = \gamma_{i1} \dots \gamma_{ir_i}$ where for a fixed i the γ_{ij} are the normalized cycles of the same length m_i and the determinant class a_i. Then the centralizer

$$C_{\Sigma_\xi(H)}(\rho) \cong C_{a_1}(\Sigma_{\xi_1}(C_{a_1})) \times C_{a_2}(\Sigma_{\xi_2}(C_{a_2})) \times \dots \times C_{a_l}(\Sigma_{\xi_l}(C_{a_l}))$$

where C_{a_i} is the centralizer of a single element $\gamma_{ij} \in \Sigma_{m_i}(H)$.
The group C_{a_i} consists of elements of the form $\kappa = [c_i]\gamma_{i1}^j$ where the element c_i belongs to the group $C_H(a_i)$ and $Char(\xi_i) = \frac{Char(\xi)}{n_k} r_i$.

Centralizers

Observe that homogenous monomial group over H becomes homogenous symmetric group when $H = \{1\}$.

Therefore our results are compatible with centralizers of elements in homogenous symmetric groups.

Centralizers of elements in homogenous symmetric groups is studied and the following is proved:

Centralizers

Theorem 5 (Güven, Kegel, Kuzucuoğlu [1])

Let ξ be an infinite sequence, $g \in S(\xi)$ and the type of principal beginning $g_0 \in S_{n_k}$ be $t(g_0) = (r_1, r_2, \ldots, r_{n_k})$. Then

$$C_{S(\xi)}(g) \cong \overset{n_k}{\underset{i=1}{Dr}} \; C_i(C_i \bar{\wr} \; S(\xi_i))$$

where $Char(\xi_i) = \frac{Char(\xi)}{n_k} r_i \quad$ *for* $i = 1, \ldots, n_k$.
If $r_i = 0$, *then we assume that corresponding factor is* $\{1\}$.

Kroshko-Sushchansky studied the diagonal type of embeddings and they give a complete characterization of such groups using Steinitz numbers.

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Kroshko N. V.; Sushchansky V. I.; Direct Limits of symmetric and alternating groups with strictly diagonal embeddings, Arch. Math. **71**, 173–182, (1998).

Theorem 6

(Kuroshko-Sushchansky, 1998)
Two groups $S(\xi_1)$ and $S(\xi_2)$ are isomorphic if and only if $Char(S(\xi_1)) = Char(S(\xi_2))$.

By using this theorem we prove the following:

Theorem 7

Let H be any finite group. The groups $\Sigma_{\xi_1}(H)$ and $\Sigma_{\xi_2}(H)$ are isomorphic if and only if $Char(\xi_1) = Char(\xi_2)$

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Thank You