ON INFLUENCE OF SUBGROUP FAMILIES, WHICH HAVE SOME FINITE RANKS, ON THE GROUP'S STRUCTURE

L.A. Kurdachenko

Ischia, 2016

There are some important numerical invariants linked to the groups. Some of them, in varying degrees, are analogous to the concept of the dimension of a vector space. The first of them is $\mathbf{d}(G)$, the minimal number of generators of a finitely generated group G. The finitely generated groups are also connected the growth functions. M. Gromov has proved the following nice structural result related to growth functions:

A finitely generated group G has polynomial growth if and only if G is nilpotent – by – finite.

On the other hand, let consider a group $G = A\langle g \rangle$ where $A \cong \mathbf{Q}_p$ and $a^g = a^p$ for every $a \in A$. This group has very simple structure and has exponential growth. Furthermore, R.I. Grigorchuk has constructed a group of intermediate growth, whose structure is very complicated. It shows that the connection between the growth functions and the structure of groups can be very weak.

The dimension of a vector space has the following good property: If B is a subspace of a vector space A, then $\mathbf{dim}_F(B) \leq \mathbf{dim}_F(A)$.

This property serves as a basis for the following notion of ***special rank***.

We say that a group G has finite special rank $\mathbf{r}(G) = \mathbf{r}$ if every finitely generated subgroup of G can be generated by at most $\mathbf{r}$ elements and there exists a finitely generated subgroup generated exactly by $\mathbf{r}$ elements.

The concept of special rank has been introduced by A.I. Maltsev (1948)

As can be seen from the definition, the concept of special rank seems to be a natural analog of the vector space dimension concept. Therefore, it is not surprising that it has been proven to be very popular and very useful. There is a huge array of articles examining the properties of groups of finite special rank, their relationship, and their influence on the structure of the group. The most general result about the groups having finite rank is the following.

We recall that a group G is said to be ***locally graded*** if every finitely generated subgroup of G has a proper subgroup of finite index.

The notion of locally graded groups has been introduced by S.N. Chernikov (1970)

Let $\mathfrak{X}$ be a class of groups. Put

$\mathbf{L}\mathfrak{X} = \{ G \mid$ every finitely generated subgroup of G belong to $\mathfrak{X} \}$,

$\mathbf{R}\mathfrak{X} = \{ G \mid$ for every element $1 \neq x \in G$ there exists a normal subgroup H_g such that $x \notin H_g$ and $G/H_g \in \mathfrak{X} \}$,

$\mathbf{P}^{\Uparrow}_{\mathbf{n}}\, \mathfrak{X} = \{\, G \mid G$ has an ascending series of normal subgroups whose factors belong to $\mathfrak{X}\, \}$,

$\mathbf{P}^{\Downarrow}_{\mathbf{n}}\, \mathfrak{X} = \{\, G \mid G$ has a descending series of normal subgroups whose factors belong to $\mathfrak{X}\, \}$.

Let $\mathfrak{T}$ be a class of periodic locally graded groups. Put $\mathfrak{W} = < \mathbf{L}, \mathbf{R}, \mathbf{P}^{\Uparrow}_{\mathbf{n}}, \mathbf{P}^{\Downarrow}_{\mathbf{n}} > \mathfrak{T}$.

We say that a group G is called ***weakly locally graded*** if $G \in \mathfrak{W}$.

N.S. Chernikov proved that the weakly locally graded groups of finite special rank are almost hyperabelian (1990).

Thus the structure of weakly locally graded groups of finite special rank is following.

We recall that a group G is said to have ***0 – rank*** $r_0(G) = r$ if G has a finite subnormal series whose factors are either infinite cyclic or periodic and if the number of infinite cyclic factors is exactly **r**.

In some papers, 0 – rank is also called the ***torsion – free rank*** of the group G.

THEOREM SR . ***Let G be a weakly locally graded group of finite special rank r. Then G has normal subgroups $V \le D$ such that V is hypercentral, D/V is abelian and G/D is finite. Moreover $\mathbf{Tor}(V)$ is a direct product of its Chernikov Sylow p – subgroups, $V/\mathbf{Tor}(V)$ is nilpotent, $D/\mathbf{Tor}(V)$ has finite 0 – rank at most r. In particular, G has finite 0 – rank. Moreover, $r_0(G) \le r$.***

We note that in general the structure of groups of finite special rank can be considerably more complicated. Using some Ol'shanskij's construction, V.N. Obraztsov (1989) has constructed an example of an uncountable group satisfying the minimal condition for all subgroups. It is possible to prove that this group has finite special rank. Thus

There exist an uncountable p – group G of finite special rank, where p is a prime such that $p \geq 10^{75}$.

But we will focus on the following question.

Let G be a group. For what family $\mathfrak{L}$ of subgroups the fact that every subgroup $L \in \mathfrak{L}$ has finite special rank implies that G has finite special rank?

We do not have the opportunity to speak here about all the results of studying the influence of various systems of subgroups of finite rank, which were obtained in the works of M.R. Dixon, M. Evans, H. Smith, F. de Giovanni, M. de Falco, C. Muzella and other authors. We will mention only a few, which are associated with our results.

M.R. Dixon, M.J. Evans and H. Smith studied the influence of locally soluble subgroups on the structure of locally (soluble – by – finite) groups. They obtained the following result (1996).

1.1. THEOREM. ***Let G be a locally (soluble – by – finite) group. If every locally soluble subgroup of G has finite special rank, then G has finite special rank.***

We want to show the generalization of this and other results, which have been obtained recently by L.A. Kurdachenko, J. Otal and I.Ya. Subbotin.

A group G is called ***generalized radical*** if G has an ascending series whose factors are locally nilpotent or locally finite.

It easily follows from its definition that a generalized radical group G either having an ascendant locally nilpotent subgroup or an ascendant locally finite subgroup. In the first case, the locally nilpotent radical of G is non – identity. In the second case, G contains a non-identity normal locally finite subgroup, so the maximal normal locally finite subgroup (locally finite radical) of G is non-identity. Thus every generalized radical group has an ascending series of normal subgroups with locally nilpotent or locally finite factors. Therefore every generalized radical group is hyper (locally nilpotent or locally finite). We also recall that a periodic locally generalized radical group is locally finite.

We want to show not only the final result, but also some of the intermediates.

1.2. THEOREM. ***Let G be a locally generalized radical group. If every locally radical subgroup of G has finite special rank, then G has finite special rank.***

1.3. COROLLARY. ***Let G be a generalized radical group. If every abelian subgroup of G has finite special rank, then G has finite special rank.***

A group G is called ***locally (soluble and minimax)*** if every finitely generated subgroup of G is soluble and minimax.

1.4. THEOREM. ***Let G be a locally generalized radical group. If every locally (soluble and minimax) subgroup of G has finite special rank, then G has finite special rank.***

1.5. COROLLARY. ***Let G be a locally generalized radical group. If every locally soluble subgroup of G has finite special rank, then G has finite special rank.***

The next our result is connected with a family of abelian subgroups.

For our purpose will be useful to recall some old classical results. The first result here was obtained by A.I. Maltsev and S.N. Chernikov

1.6. THEOREM . ***Let G be a locally nilpotent group. If every abelian subgroup of G has finite special rank, then G has finite special rank.***

Soluble case was considered by M.I. Kargapolov (1962).

1.7. THEOREM . ***Let G be a soluble group. If every abelian subgroup of G has finite special rank, then G has finite special rank.***

1.8. COROLLARY . ***Let G be a radical group. If every abelian subgroup of G has finite special rank, then G has finite special rank.***

However, for locally soluble groups the situation is more complicated. Yu.I. Merzlyakov proved the following result (1969).

1.9. THEOREM . ***There exists a locally polycyclic torsion – free group G having infinite special rank, whose abelian subgroups have finite special rank.***

But for the periodic locally soluble groups the situation is different. The following result which has been obtained by Yu.M. Gorchakov (1964) justifies it.

1.10. THEOREM . ***Let G be a periodic locally soluble group. If every abelian subgroup of G has finite special rank, then G has finite special rank.***

V. P. Shunkov (1971) extended this result on locally finite groups.

As we can see from **Corollary 1.3** the impact of the system of abelian subgroups in generalized radical groups is very important, while it is much weaker in locally soluble groups. Figuratively speaking, the border runs between radical groups and locally soluble groups. Now we tried to delineate this border more precisely. We base ourselves on the observation that in the group constructed by Y.I. Merzlyakov the ranks of chief factors are not bounded.

Let $\mathbf{b}$ be a positive integer. A group G is called ***b – generalized radical*** if G has an ascending series of normal subgroups whose factors are locally nilpotent or locally finite groups of special rank $\mathbf{b}$.

1.11. THEOREM. ***Let b be a positive integer and G be a locally b – generalized radical group. If every abelian subgroup of G has finite special rank, then G has finite special rank.***

1.12. COROLLARY. ***Let b be a positive integer and G be a group. Suppose that G has an ascending series whose factors are locally b – generalized radical group. If every abelian subgroup of G has finite special rank, then G has finite special rank.***

Let us now consider some analogues of the above results for the other rank.

Let p be a prime. We say that a group G has ***finite section p – rank*** $\mathbf{sr}_p(G) = \mathbf{r}$ if every elementary abelian p – section of G is finite of order at most $p^{\mathbf{r}}$, and there is an elementary abelian p – section A/B of G such that $|A/B| = p^{\mathbf{r}}$.

We say that a group G has finite ***section rank*** if $\mathbf{sr}_p(G)$ is finite for each prime number p.

We can slightly concretize this definition. Let σ be a function from the set $\mathbf{P}$ of all primes in $\mathbf{N}_0$. We say that a group G has finite ***section rank*** $\boldsymbol{\sigma}$, if $\mathbf{sr}_p(G) = \sigma(p)$ for every prime p.

A group G is said to be a group of ***finite abelian section rank*** if every elementary abelian section of G is finite (D.J.S. Robinson).

We note that if G has finite section rank, then G is a group of finite abelian section rank. But converse is not true. If G is a group constructed by Yu.I. Merzlyakov, then every elementary abelian section of G is finite, but for every prime p, the orders of elementary abelian p – sections of G are not bounded.

A group G is called ***nearly radical*** if G has an ascending series whose factors are locally nilpotent or finite. R. Baer and H. Heineken used the term generalized radical groups for such groups, but we have used it for a much wider class of groups.

1.13. THEOREM. ***Let G be a locally nearly radical group. If every locally (soluble and minimax) subgroup of G has finite section rank, then G is almost radical and has finite section rank.***

1.14. COROLLARY. ***Let G be a locally (soluble – by – finite) group. If every locally (soluble and minimax) subgroup of G has finite section rank, then G is almost radical and has finite section rank.***

1.15. COROLLARY. ***Let G be a locally (soluble – by – finite) group. If every locally soluble subgroup of G has finite section rank, then G is almost radical and has finite section rank.***

For the next result will be useful to recall the following result obtained by Yu.I. Merzlyakov (1964).

1.16. THEOREM . ***Let G be a locally soluble group. If there exists a positive integer b such that every abelian subgroup of G has finite special rank at most b, then G has finite special rank, moreover it is bounded by some function of b.***

Comparison this theorem and **Theorem 1.10** leads to the idea that the infinity of the special rank of a group is due unboundedness of special ranks of torsion - free abelian subgroups. M.R. Dixon, M. Evans and H. Smith (1996) obtained the next result, combining a sense of these theorems.

1.17. THEOREM. ***Let G be a locally (soluble – by – finite) group. If every abelian subgroup of G has finite special rank and there exists a positive integer b such that $r_0(A) \leq b$ for every abelian subgroup A of G, then G has finite special rank.***

We have extended this result to a larger class and got its counterpart for section rank.

1.18. THEOREM. ***Let G be a locally nearly radical group. Suppose that there exists a positive integer b such that $r_0(A) \leq b$ for every abelian subgroup A of G. If all abelian subgroups of G have finite section rank, then G has finite section rank.***

1.19. COROLLARY. ***Let G be a locally (soluble – by – finite) group. Suppose that there exists a positive integer b such that $r_0(A) \leq b$ for every abelian subgroup A of G. If every abelian subgroup of G has finite section rank, then G has finite section rank.***

1.20. COROLLARY. ***Let G be a locally generalized radical group. Suppose that there exists a positive integer b such that $r_0(A) \leq b$ for every abelian subgroup A of G. If every abelian subgroup of G has finite special rank, then G has finite special rank.***

In conclusion we observe that the impact of the family of locally soluble subgroups in another types of groups can also be very weak. Thus, if G is a free group then all locally soluble subgroups of G have special rank 1, but the special rank of G is infinite.