Generators for discrete subgroups of 2-by-2 matrices over rational quaternion algebras

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Goal: Generators for $\mathrm{SL}_2(\mathcal{H}(\frac{a,b}{\mathbb{Z}}))$

- consider $M_2(\mathcal{H}(\frac{a,b}{\mathbb{Q}}))$ with $a, b < 0$
- order $\mathcal{H}(\frac{a,b}{\mathbb{Z}})$ in $\mathcal{H}(\frac{a,b}{\mathbb{Q}})$

Goal of this work

Finding generators for $\mathrm{SL}_2(\mathcal{H}(\frac{a,b}{\mathbb{Z}}))$.

Motivation: Units in Group Rings

- G finite group.

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- $\mathbb{Q}G = \prod_{i=1}^{n} M_{n_i}(D_i)$, D_i a division algebra,
- let $\mathcal{O}_i$ be an order in D_i for every $1 \leq i \leq n$.

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Finding generators and relations for $\mathcal{U}(\mathbb{Z}G)$, up to commensurability, reduces to finding generators and relations for $\mathrm{SL}_{n_i}(\mathcal{O}_i)$ for every $1 \leq i \leq n$.

Unsolved cases

three cases that are unsolved

- $M_2\left(\mathcal{H}\left(\frac{-1,-1}{\mathbb{Q}}\right)\right)$
- $M_2\left(\mathcal{H}\left(\frac{-1,-3}{\mathbb{Q}}\right)\right)$
- $M_2\left(\mathcal{H}\left(\frac{-2,-5}{\mathbb{Q}}\right)\right)$

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Idea: discontinuous actions on hyperbolic space

Isometries of $\mathbb{H}^2$ and $\mathbb{H}^3$

The upper half space model of hyperbolic space

- $\mathbb{H}^2 = \{z = x + yi \mid x, y \in \mathbb{R}, y > 0\}$
- $\mathbb{H}^3 = \{z = x + yi + rj \mid x, y, r \in \mathbb{R}, r > 0\}$

- $\mathrm{PSL}_2(\mathbb{R}) \cong \mathrm{ISO}^+(\mathbb{H}^2)$
- $\mathrm{PSL}_2(\mathbb{C}) \cong \mathrm{ISO}^+(\mathbb{H}^3)$

Action on $\mathbb{H}^2, \mathbb{H}^3$

- $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}, \text{ computed in } \mathbb{C}$
- $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = (az+b)(cz+d)^{-1}, \text{ computed in } \mathcal{H}(\frac{-1,-1}{\mathbb{R}})$

Group Actions, Fundamental Domains and Poincaré

Theorem

Let X be a proper metric space. A group Γ of isometries of X acts discontinuously on X if and only if it is a discrete subgroup.

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Definition

A fundamental domain of the discontinuous group $\Gamma < \mathrm{Iso}(X)$ is a closed subset $\mathcal{F} \subseteq X$ satisfying the following conditions:

- the boundary of $\mathcal{F}$ has Lebesgue measure 0,
- $g(\mathcal{F}^\circ) \neq h(\mathcal{F}^\circ)$ for $g \neq h$.
- $X = \bigcup_{g \in \Gamma} g(\mathcal{F})$.

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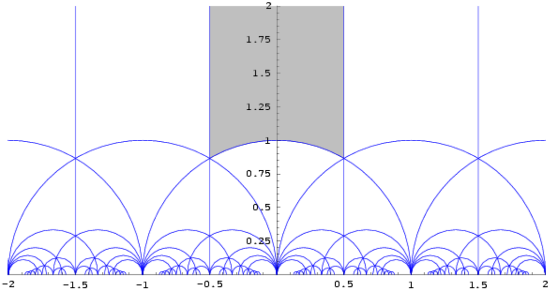
Theorem (Poincaré)

Let $\mathcal{F}$ be a convex fundamental polyhedron for a discrete group Γ of $\mathbb{H}^n$. Then Γ is generated by

$$\{g \in \Gamma \mid \mathcal{F} \cap g(\mathcal{F}) \text{ is a side of } \mathcal{F}\}.$$

Fundamental Domain of $\mathrm{PSL}_2(\mathbb{Z})$

Fundamental Domain of $\mathrm{PSL}_2(\mathbb{Z})$ acting on $\mathbb{H}^2$



Dirichlet Algorithm of Finite Covolume (DAFC)

joint work with E. Jespers, S. O. Juriaans, A. De A. E Silva, A. C. Souza Filho

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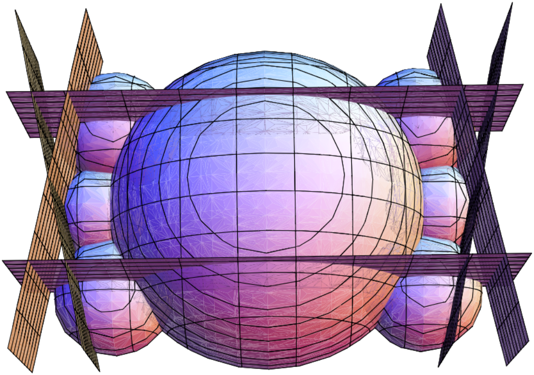
$\Gamma \leq \mathrm{PSL}_2(\mathbb{C})$ discrete group of finite covolume

$\longrightarrow$

- finite-sided convex polyhedron P of finite volume
- P contains fundamental domain for Γ
- finite set of generators up to finite index for Γ

Output for $\mathrm{PSL}_2\left(\mathbb{Z}\left[\frac{1+\sqrt{-23}}{2}\right]\right)$



What about $M_2(\mathcal{H}(\frac{a,b}{\mathbb{Q}}))$?

- to begin: $M_2(\mathcal{H}(\frac{-1,-1}{\mathbb{Q}}))$
- order: $\mathrm{PSL}_2(\mathcal{H}(\frac{-1,-1}{\mathbb{Z}}))$

Main idea: imitate DAFC for $\Gamma \leq \mathrm{PSL}_2(\mathcal{H}(\frac{-1,-1}{\mathbb{R}}))$ discrete

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What is $\mathrm{PSL}_2(\mathcal{H}(\frac{-1,-1}{\mathbb{R}}))$?

$\rightarrow$ reduced norm 1

The action of $\mathrm{PSL}_2(\mathcal{H}(\frac{-1,-1}{\mathbb{R}}))$

- Möbius action on $\mathcal{H}(\frac{-1,-1}{\mathbb{R}})$
- action on $\mathbb{H}^5$ by Poincaré extension

$$\rightarrow \mathrm{PSL}_2(\mathcal{H}(\tfrac{-1,-1}{\mathbb{R}})) \cong \mathrm{ISO}^+(\mathbb{H}^5).$$

However: not very handy to work with.

Clifford Algebras

Definition

The Clifford algebra $\mathcal{C}_n$ is the associative algebra over the reals generated by elements $i_1, i_2, \ldots i_{n-1}$ satisfying

- $i_h^2 = -1$ for every $1 \leq h \leq n-1$
- $i_h i_l = -i_l i_h$ for $h \neq l$.

The Clifford vector space $\mathbb{V}^n$

vector space of all Clifford elements of the form
$a_0 + a_1 i_1 + \ldots + a_{n-1} i_{n-1}$.
If $a \in \mathbb{V}^n$, then $|a|^2 = a\bar{a}$

$\rightarrow$ every non-zero vector is invertible

$\rightarrow$ the product of non-zero vectors is invertible

Definition

The Clifford group Γ_n is the group of all products of non-zero vectors.

$\mathrm{PSL}_+(\Gamma_n)$ and its action on $\mathbb{H}^{n+1}$

$$\mathrm{SL}_+(\Gamma_n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad^* - bc^* = 1, ab^*, cd^*, c^*a, d^*b \in \mathbb{V}^n \right\}$$

Theorem (Ahlfors '81)

$\mathrm{PSL}_+(\Gamma_n) \cong \mathrm{ISO}^+(\mathbb{H}^{n+1})$.

Möbius action

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = (az+b)(cz+d)^{-1}$, computed in $\mathcal{C}_{n+1}$

Main strategy: Imitate the DAFC for $\mathrm{PSL}_+(\Gamma_4(\mathbb{Z})) \leq \mathrm{PSL}_+(\Gamma_4)$, discrete.

To sum up

$$\mathrm{PSL}_2(\mathcal{H}(\frac{-1,-1}{\mathbb{R}})) \cong \mathrm{ISO}^+(\mathbb{H}^5) \cong \mathrm{PSL}_+(\Gamma_4)$$

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$$\mathcal{H}(\frac{-1,-3}{\mathbb{R}}) \cong \mathcal{H}(\frac{-1,-1}{\mathbb{R}}) \cong \mathcal{H}(\frac{-2,-5}{\mathbb{R}})$$

Thank you for your attention.