Engel-type subgroups in finite and profinite groups

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Joint work with Pavel Shumyatsky

Engel groups

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J. Wilson and E. Zelmanov, 1992

proved the converse for profinite groups:
any Engel profinite group is locally nilpotent.

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Remark: Note that this is not a subnormal subgroup, unlike the subgroups

$$G \trianglerighteq [G, g] \trianglerighteq [[G, g], g] \trianglerighteq \cdots$$

Almost Engel profinite groups

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Theorem 1

Suppose that G is a profinite group such that for every $g \in G$ there is a positive integer $n = n(g)$ such that $E_n(g)$ is finite.

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It also follows that there is an open locally nilpotent subgroup (just consider $C_G(N)$) — but this fact is actually one of the steps in the proof.

Finite groups: notation

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Notation: The nilpotent residual of a group G is

$$\gamma_\infty(G) = \bigcap_i \gamma_i(G),$$

where $\gamma_i(G)$ are terms of the lower central series

$(\gamma_1(G) = G$, and $\gamma_{i+1}(G) = [\gamma_i(G), G])$.

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Remark: Theorem 2 can be viewed as a generalization of Zorn's theorem that a finite Engel group is nilpotent.

Quantitative version for profinite groups

This theorem implies a similar result for profinite groups.

Corollary

Suppose that G is a profinite group and there is a positive integer m such that for every $g \in G$ there is $n = n(g)$ such that $|E_n(g)| \leqslant m$.

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Suppose that G is a profinite group and there is a positive integer m such that for every $g \in G$ there is $n = n(g)$ such that $|E_n(g)| \leqslant m$. Then G has a finite normal subgroup N of order bounded in terms of m such that G/N is locally nilpotent.

Engel-type subgroups in length results: soluble groups

By Baer's theorem, any Engel element of a finite group belongs to its Fitting subgroup: if $[x, g, g, \dots, g] = 1$ for all $x \in G$, then $g \in F(G)$.

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Theorem 3

If g is an element of a soluble finite group G such that $E_n(g)$ (for some n) has Fitting height k, then $g \in F_{k+1}(G)$.

Engel-type subgroups in relation to generalized Fitting height

The generalized Fitting height $h^*(G)$ of a finite group G is the least h such that $F_h^*(G) = G$, where $F_1^*(G) = F^*(G)$ is the generalized Fitting subgroup, and by induction $F_{i+1}^*(G)/F_i^*(G) = F^*(G/F_i^*(G))$.

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Theorem 4

If g is an element of a finite group G such that $E_n(g)$ (for some n) has generalized Fitting height k, then $g \in F_{f(k,m)}^*(G)$, where m is the number of prime divisors of $|g|$.

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The nonsoluble length $\lambda(G)$ of a finite group G is defined as the minimum number of nonsoluble factors in a normal series each of whose factors either is soluble or is a direct product of nonabelian simple groups.

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Reduction of conjectures

Question

Let $S = S_1 \times \cdots \times S_r$ be a direct product of nonabelian finite simple groups, and φ an automorphism of S transitively permuting the factors. Is it true that $E_n(\varphi) = S$ for any n?

Here $E_n(\varphi) = \langle [g, \underbrace{\varphi, \varphi, \ldots, \varphi}_{n}] \mid g \in S \rangle$.

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Some progress was made for the Question in the case where $|\varphi|$ is a prime by Robert Guralnick (unpublished).