On the nonabelian tensor product of cyclic groups of p-power order

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History

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R. Brown, J.-L. Loday, Exision homotopique en basse dimension, C.R. Acad. Sci. Ser. I Math. Paris 298 (1984), 353-356.

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Definition. Let G and H be groups which act on each other via automorphisms and which act on themselves via conjugation. The actions are said to be compatible if

$$ {}^{{}^{g}h}g' = {}^{g}({}^{h}({}^{g^{-1}}g')) \text{ and } {}^{{}^{h}g}h' = {}^{h}({}^{g}({}^{h^{-1}}h')) $$

for all $g, g' \in G$ and $h, h' \in H$.

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for $g, g' \in G$ and $h, h' \in H$.
If $G = H$ and the actions are conjugation, which are always compatible, we call $G \otimes G$ the nonabelian tensor square of G.

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Theorem 1. *For a group G the following conditions are equivalent:*

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(v) $G \otimes G \cong G/G' \otimes G/G'$.

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Example 1. Let $G \cong H \cong C_0$, the infinite cyclic group, with mutual action being the inversion. Then the actions are compatible and $C_0 \otimes C_0 \cong C_0 \oplus C_0$.

Theorem 3. *Let C_n, C_m be cyclic groups of order n, and m, respectively, with $n, m \geq 0$, acting on each other compatibly. Then $C_n \otimes C_m$ has at most 2 generators.*

Proposition 1. *Let* $G = \langle x \rangle \cong C_m$*,* $H = \langle y \rangle \cong C_n$ *and* ${}^{y}x = x^k$ *and* ${}^{x}y = y$*. Then the actions are compatible and*
$G \otimes H = \langle x \otimes y \rangle$ *with* $|x \otimes y| = gcd\left(m, \dfrac{k^n - 1}{k - 1}\right)$.

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Proposition 2. *Let* $G = \langle x \rangle \cong C_m$ *and* $H = \langle y \rangle \cong C_n$*,* $n, m \geq 0$ *and* $2|m$*,* $2|n$ *with* ${}^{x^2}y = y$ *and* ${}^{y^2}x = x$*. Then the actions are compatible.*

Theorem 4. *Let* $G = \langle x \rangle \cong C_m$ *and* $H = \langle y \rangle \cong C_n$*. Furthermore, let* $\sigma : H \to Aut(G)$ *and* $\tau : G \to Aut(H)$ *be actions, where* H *acts on* G *and* G *acts on* H*, respectively, such that*

$$\sigma : \ {}^{y}x = x^{s} \ \text{ and } \ \tau : \ {}^{x}y = y^{t},$$

where s *and* t *are positive integers.*

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$$\sigma : \ {}^{y}x = x^{s} \quad \text{and} \quad \tau : \ {}^{x}y = y^{t},$$

where s *and* t *are positive integers. Then the actions are compatible if and only if* $s \equiv 1 \bmod |\sigma|$ *and* $t \equiv 1 \bmod |\tau|$*.*

Example 2. Let $C_3 = \langle x \rangle$, $C_2 = \langle y \rangle$ and $C_7 = \langle z \rangle$ be cyclic groups of prime power order.

The following mappings on the generators

$$^{y}x = x^2,\ ^{z}x = x,\ ^{x}y = y,\ ^{x}z = z^4;$$

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extend linearly to actions. The resulting mutual actions between C_3 and C_2 are compatible as well as those between C_3 and C_7. However, the induced mutual actions between C_3 and $C_2 \times C_7 \cong C_{14}$ are not compatible.

Cyclic groups of p-power order,
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Theorem 5. *Let p be an odd prime and $G = \langle g \rangle \cong C_{p^\alpha}$, $H = \langle h \rangle \cong C_{p^\beta}$, where $\alpha, \beta \geq 2$. Furthermore, let $\sigma \in Aut(G)$ with $|\sigma| = p^s$, where $1 \leq s \leq \alpha - 1$ and $\tau \in Aut(H)$ with $|\tau| = p^t$, where $1 \leq t \leq \beta - 1$.*

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Theorem 6. *Let p be an odd prime and $G = \langle g \rangle \cong C_{p^\alpha}$, $H = \langle h \rangle \cong C_{p^\alpha}$, where $\alpha, \beta \geq 2$, with the actions*

$$^{y}x = x^{ip^{\alpha-s}+1} \quad \text{and} \quad ^{x}y = y^{jp^{\beta-t}+1},$$

where $gcd(i,p) = gcd(j,p) = 1$ and $s + t \leq \min(\alpha, \beta)$.

Theorem 6. *Let p be an odd prime and $G = \langle g \rangle \cong C_{p^\alpha}$, $H = \langle h \rangle \cong C_{p^\alpha}$, where $\alpha, \beta \geq 2$, with the actions*

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where $gcd(i,p) = gcd(j,p) = 1$ and $s + t \leq \min(\alpha, \beta)$. Then $G \otimes H$ is cyclic and a homomorphic image of C_{p^γ}, where $\gamma = \min\{\alpha, \beta\}$.

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Theorem 7. *Let $G = \langle g \rangle \cong C_{2^\alpha}$ and $H = \langle h \rangle \cong C_{2^\beta}$ with $\alpha \geq 2$ and $\beta \geq 3$. Furthermore, let $\sigma \in Aut(G)$ with $|\sigma| = 2$ and $\tau \in Aut(H)$.*

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Theorem 7. *Let $G = \langle g \rangle \cong C_{2^\alpha}$ and $H = \langle h \rangle \cong C_{2^\beta}$ with $\alpha \geq 2$ and $\beta \geq 3$. Furthermore, let $\sigma \in Aut(G)$ with $|\sigma| = 2$ and $\tau \in Aut(H)$.*

(i) *If $\sigma(g) = g^s$ with $s \equiv -1 \mod 2^\alpha$ or $s \equiv 2^{m-1} - 1 \mod 2^\alpha$, then (σ, τ) is a compatible pair if and only if τ is the trivial automorphism or $|\tau| = 2$.*

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(ii) *If $\sigma(g) = g^s$ with $s = 2^{\alpha-1} + 1$, then (σ, τ) is a compatible pair if and only if $|\tau| \leq 2^t$ with $t \leq \alpha - 1$, in particular σ is compatible with all $\tau \in Aut(H)$ provided $\beta \leq \alpha + 1$.*

Theorem 8. *Let* $G = \langle g \rangle \cong C_{2^\alpha}$ *and* $H = \langle h \rangle \cong C_{2^\beta}$ *with* ${}^h g = g^s$ *and* ${}^g h = h^t$*:*

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(i) *if* $t = 2^{\beta-1} + 1$, $\beta \geq 3$, *then*

$$G \otimes H \cong \begin{cases} C_{2^\alpha}, & \text{if } s = 2^{\alpha-1} - 1 \text{ or } 2^\alpha - 1, \\ C_2^{\min(\alpha,\beta)}, & \text{if } s = 2^{\alpha-1} + 1,\ \alpha \geq 3; \end{cases}$$

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$$G \otimes H \cong \begin{cases} C_{2^{\max(\alpha,\beta)}} \times C_{2^{\min(\alpha,\beta)-1}}, & \text{if } s = 2^\alpha - 1, \\ C_{2^{\max(\alpha-1,\beta)}} \times C_{2^{\min(\alpha-1,\beta)-1}}, & \text{if } s = 2^{\alpha-1} - 1, \\ & \alpha \geq 3; \end{cases}$$

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(iii) *if* $t = 2^{\beta-1} - 1$ *and* $s = 2^{\alpha-1} - 1$, $\alpha \geq \beta \geq 3$, *then*

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Theorem 9. *Let* $G = \langle g \rangle = C_{2^\alpha}$ *and* $H = \langle h \rangle \cong C_{2^\beta}$ *and let* (σ, τ) *be a compatible pair with* $\sigma(g) = g^{2^{\alpha-1}+1}$*. Then* $G \otimes H$ *is cyclic of 2-power order.*

Theorem 10. *Let* $G = \langle g \rangle \cong C_{2^\alpha}$ *and* $H = \langle h \rangle \cong C_{2^\beta}$*. Let* $\sigma \in Aut(G)$ *with* $|\sigma| = 2^r$*,* $r \geq 2$*, and* $\tau \in Aut(H)$ *with* $\alpha \geq 4$*, and* $\beta \geq 2$*.*

Theorem 10. *Let $G = \langle g \rangle \cong C_{2^\alpha}$ and $H = \langle h \rangle \cong C_{2^\beta}$. Let $\sigma \in Aut(G)$ with $|\sigma| = 2^r$, $r \geq 2$, and $\tau \in Aut(H)$ with $\alpha \geq 4$, and $\beta \geq 2$.*

(i) *If $\sigma(g) = g^s$ with $s \equiv (-1)^i \cdot 5^j$ mod 2^α and $i = 1$, then (σ, τ) is a compatible pair if and only if $\tau(h) = h^t$ with $t \equiv 1$ mod 2^β or $t \equiv 2^{\beta-1} + 1$ mod 2^β.*

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(ii) *If $\sigma(g) = g^s$ with $s \equiv (-1)^i 5^j \bmod 2^\alpha$ and $i = 0$, then (σ, τ) is a compatible pair if and only if $|\tau| \leq 2^{\alpha-r}$ provided $\beta \leq \alpha - r + 2$.*

Theorem 11. *Let* $G = \langle g \rangle \cong C_{2^\alpha}$ *and* $H = \langle h \rangle \cong C_{2^\beta}$ *and let* (σ, τ) *be a compatible pair with* $|\sigma| = 2^r$ *and* $|\tau| = 2^q$*,* $r, q \geq 2$*. Then* $G \otimes H$ *is a homomorphic image of* C_{2^γ} *with* $\gamma = \min(\alpha, \beta)$*.*