Pro-p methods in the theory of finite p-groups

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Outline

General setting

$\mathcal{P}$ is a property of finite p-groups

General question

Is there only a finite number of finite p-groups satisfying the property $\mathcal{P}$?

Part I: The use of pro-p groups to give a positive answer.

Part II: The use of pro-p groups to give a negative answer.

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Constructing a pro-p group

$\mathcal{P}$ is a property of finite p-groups

$\mathcal{F} = \{$ finite p-groups G satisfying $\mathcal{P}\}$

By way of contradiction, we assume that $\mathcal{F}$ is infinite

Our goals are

First, we want to construct an infinite "nice" pro-p group F such that infinitely many finite quotients of F are in $\mathcal{F}$.

Second, we want to show that a such pro-p group F does not exist.

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G is a finite p-group, **the lower p-central series**:

$$\lambda_1(G) = G,\ \lambda_{i+1}(G) = [\lambda_i(G), G]\lambda_i(G)^p.$$

$\Gamma_{\mathcal{F}}$ is a directed locally finite forest:
$V(\Gamma_{\mathcal{F}}) = \{\text{isomorphism classes of non-trivial groups from } \mathcal{F}\}$,
$G_1 \to G_2$ ((G_1, G_2) is an edge) if there exists k such that

$$G_1 \cong G_2/\lambda_k(G_2), \lambda_k(G_2) \neq \{1\}, \lambda_{k+1}(G_2) = \{1\}$$

G_2 is a **son** of G_1

A **finite path**: $G_0 \to G_1 \to \ldots \to G_k$. G_k is a **descendant** of G_0.
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$G \in \mathcal{F}$, $\Gamma_G = \{\text{descendents of } G\}$

Proposition

Assume that

1. $\Gamma_{\mathcal{F}}$ is infinite and
2. $\Gamma_{\mathcal{F}}$ is a union of a finite number of Γ_G $(G \in \mathcal{F})$.

Then there exists an infinite path in Γ.

Proof: There exists G_0 such that Γ_{G_0} is infinite.
Γ_{G_1} is infinite for some son G_1 of G_0.
$G_0 \to G_1 \to \ldots \to G_k$, Γ_{G_k} is infinite.
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$G_0 \to G_1 \to G_2 \to \ldots \to G_k \to \ldots$ (an infinite path)

$F = \varprojlim G_i$ for $G_0 \leftarrow G_1 \leftarrow G_2 \leftarrow \ldots \leftarrow G_k \leftarrow \ldots$ (G_i is a quotient of G_{i+1})

Corollary

Let $\mathcal{P}$ be a property of finite p-groups. Assume that

1. There are infinitely many finite p-groups satisfying $\mathcal{P}$.
2. There exists a constant k such that if G satisfies $\mathcal{P}$ then $G/\lambda_i(G)$ satisfies $\mathcal{P}$ for every $i \geq k$.
3. There exists a constant d such that any group, that satisfies $\mathcal{P}$, is d-generated.

Then there exists an infinite finitely generated pro-p group F such that for every $i \geq k$, $F/\lambda_i(F)$ satisfies $\mathcal{P}$.

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Previous applications: finite p-groups of bounded coclass

The **coclass** $cc(G)$ of a finite p-group G of order p^m and nilpotency class $c(G)$ is the difference $cc(G) = m - c(G)$.

Therem (C. Leedham-Green, A. Shalev + many other mathematicians)

There exists a function $f(p, r)$ such that every finite p-group of coclass at most r contains a subgroup of class at most 2 (abelian if $p = 2$) and index at most $f(p, r)$.

C. Leedham-Green: gave a proof using pro-p methods.
A. Shalev: gave an effective proof.

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Previous applications: almost regular automorphisms of finite p-groups

Theorem (A. Jaikin-Zapirain)

There exists a function $f(p, m, n)$ such that every finite p-group, admitting an automorphism of order p^n with p^m fixed points, has a subgroup of index $f(p, m, n)$ and derived length at most $2^{m+1} - 2$.

Previous applications: restrictions on the set of character degrees of finite p-groups

Theorem (A. Jaikin-Zapirain, A. Moreto)

Let $\mathcal{S}$ be a set of powers of p, containing 1. Let

$$p^k = \min\{x \in \mathcal{S} : \ x \neq 1\} > p.$$

Assume that

$$|\mathcal{S}| \leq \begin{cases} k+1 & \text{if } p = 2 \\ k+2 & \text{if } p > 2 \end{cases}.$$

Then there exists a constant $C_{\mathcal{S}}$ such that if the complex character degrees of a finite p-group G belong to $\mathcal{S}$, then $c(G) \leq C_{\mathcal{S}}$.

The range of applications of pro-p methods

When the pro-p methods can be applied?

In the three previous situations the pro-p groups that appear are p-adic analytic.

The Restricted Burnside Problem (solved by E. Zelmanov)

There are only finitely many d-generated finite p-groups of exponent p^m.

It is equivalent to

There are no infinite finitely generated pro-p groups of finite exponent.

Thus, the pro-p methods can be applied but they do not help.

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p-adic analytic groups

A pro-p-group P is p-**adic analytic** if one of the following equivalent conditions holds

1. P has a structure of a p-adic manifold and the group operation are analytic functions with respect to this structure.
2. P is a closed subgroup of $\mathrm{GL}_n(\mathbb{Z}_p)$ for some n.
3. P has finite **rank** (the number of generators of closed subgroups is uniformly bounded).
4. P contains an open **uniform** pro-p subgroup U (U is torsion free, finitely generated and $[U, U] \le U^{2p}$).

$\dim P = \dim_{\mathbb{Q}_p} \mathcal{L}(P)$, where $\mathcal{L}(P)$ is the associated Lie algebra

$\dim U = \log_p |U : U^p|$ for U uniform

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2. P is a closed subgroup of $\mathrm{GL}_n(\mathbb{Z}_p)$ for some n.
3. P has finite **rank** (the number of generators of closed subgroups is uniformly bounded).
4. P contains an open **uniform** pro-p subgroup U (U is torsion free, finitely generated and $[U, U] \le U^{2p}$).

$\dim P = \dim_{\mathbb{Q}_p} \mathcal{L}(P)$, where $\mathcal{L}(P)$ is the associated Lie algebra

$\dim U = \log_p |U : U^p|$ for U uniform

2-groups with odd number of conjugacy classes

A conjugacy class of $x \in G$ is called **real** if x is conjugate to its inverse.

The parity of the number of conjugacy classes of a finite group coincides with the parity of the number of its real classes.

The most small 2-groups have an even number conjugacy classes.

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Let r be an odd natural number. Then there are only finitely many finite 2-groups with r real conjugacy classes.

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Proposition

Let r be a natural number. Assume that a finite 2-group G has exactly r real conjugacy classes.

1. There exists a constant k (depending only on r) such that $G/\lambda_k(G)$ has exactly r real conjugacy classes.
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1. F is 2-adic analytic;
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3. if r is odd, then $F/rad_f(F)$ has also odd number $(\leq r)$ of real irreducible characters;
4. if r is odd, then any just infinite quotient of F has odd number $(\leq r)$ of real irreducible characters and it is not solvable.

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Corollary

We can assume that F is a torsion free, non-solvable, just infinite 2-adic analytic pro-2 group with r real irreducible characters ($r < 24$ is an odd number).

A non-solvable just infinite p-adic analytic pro-p group is isomorphic to an open subgroups of a Sylow pro-p subgroup of the automorphism group of a finite dimensional semisimple p-adic Lie algebra.

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Proposition (an application of Knesser's classification)

Let $\mathcal{L}$ be a finite-dimensional semi-simple Lie $\mathbb{Q}_2$-algebra. Assume that the Sylow pro-2 subgroups of $\mathrm{Aut}(\mathcal{L})$ are torsion free and just infinite. Then $\mathcal{L}$ is isomorphic to $\mathfrak{sl}_1(D)$ for some finite-dimensional division $\mathbb{Q}_2$-algebra D.

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Let $\mathcal{L} = \mathfrak{sl}_1(D)$ for some finite-dimensional division $\mathbb{Q}_2$-algebra D. Then a Sylow pro-2 subgroup of $\mathrm{Aut}(\mathcal{L})$ can not have odd number of real irreducible characters smaller than 24.

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Let r be an odd natural number less than 24. Then there are only finitely many finite 2-groups with r real conjugacy classes.

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2. Part I: a positive answer
3. Part II: a negative answer

Pro-p group as a source of counterexamples

$\mathcal{P}$ is a property of finite p-groups

General question

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How to show that the answer is NO?

Our goal is to construct an infinite pro-p group F such that almost all finite quotients of F satisfies the property $\mathcal{P}$

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Problem

Is it true that $|G|$ divides $|\mathrm{Aut}(G)|$ for every non-abelian finite p-group G.

E. Schenkman, 1955

Known to be true for many families of p-groups

Theorem (J. González-Sánchez, A. Jaikin-Zapirain)

For each prime p there exists an infinite family of finite p-groups $\{U_i\}$ such that

$$|\mathrm{Aut}\, U_i| \le O(|U_i|^{\frac{40}{41}})\ (i \to \infty).$$

In particular, for every prime p, there exists a non-abelian finite p-group G such that $|\mathrm{Aut}(G)| < |G|$?

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An explanation of the construction:

Our fist aim is to find a pro-p group U such that $\mathrm{Aut}(U)$ is "smaller" than U.

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Sato's example: a Lie $\mathbb{Q}$-algebra L, $\dim_{\mathbb{Q}} L = 41$, $\dim_{\mathbb{Q}} Z(L) = 1$, $\mathrm{Der}(L) = \mathrm{Inn}(L)$.

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Sato's example: a Lie $\mathbb{Q}$-algebra L, $\dim_{\mathbb{Q}} L = 41$, $\dim_{\mathbb{Q}} Z(L) = 1$, $\mathrm{Der}(L) = \mathrm{Inn}(L)$.

Let M be a Lie $\mathbb{Z}$-algebra such that $L = \mathbb{Q} \otimes_{\mathbb{Z}} M$

$U = \exp(M_p)$ ($M_p = 2p(\mathbb{Z}_p \otimes_{\mathbb{Z}} M)$) is a uniform pro-$p$ Lie ring)

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$|\mathrm{Aut}(U_i)|$ **is smaller than** $|U_i|$ **when** i **is large:**

Proposition

Let U be a uniform pro-p group and assume that $\mathcal{L}(U)$ has only inner derivations. Then there exists a constant C such that

$$|\mathrm{Aut}(U/U^{p^i}) : \mathrm{Inn}(U/U^{p^i})| \leq C.$$

In our examples:
$|U/U^{p^i}| = p^{41i}$, $Z(U/U^{p^i})| \geq p^i \Rightarrow |\mathrm{Inn}(U/U^{p^i})| \leq p^{40i}$.

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Thanks

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