Talk in Ischia – 2016

Recent generalizations of the Berkovich-Chillag-Herzog theorem

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Introduction.

All groups in this talk are finite. In 1968, Gary Seitz proved the following theorem:

Theorem 1

The group G has exactly one non-linear irreducible character if and only if G is of one of the following two types:

(a) *G is an extra-special 2-group of order 2^{2m+1}. The degree pattern of G is $(1^{(2^m)}, 2^m)$;*

(b) *G is a doubly transitive Frobenius group of order $(p^n - 1)p^n$ with a cyclic complement. The degree pattern of G is $(1^{(p^n-1)}, p^n - 1)$.*

A Frobenius group G is called *doubly transitive* if

$$|H| = |F| - 1,$$

where F denotes the (Frobenius) kernel of G and H denotes a (Frobenius) complement of G. This implies that

$$|G| = (p^n - 1)p^n \quad \text{with} \quad |F| = p^n \text{ and } |H| = p^n - 1,$$

where p is a prime and n is a positive integer. Moreover, F is an elementary abelian p-group. We shall use this notation throughout this talk.

In 1992, Yakov Berkovich, David Chillag and Marcel Herzog generalized Seitz's theorem. They proved

Theorem 2

The non-linear irreducible characters of the non-abelian group G are of distinct degrees if and only if either G is of one of the two types obtained in Theorem 1:

(a) *G is an extra-special 2-group of order 2^{2m+1}. The degree pattern of G is $(1^{(2^m)}, 2^m)$;*

(b) *G is a doubly transitive Frobenius group of order $(p^n-1)p^n$ with a cyclic complement. The degree pattern of G is $(1^{(p^n-1)}, p^n-1)$;*

or G is the following group:

(c) *G is a doubly transitive Frobenius group of order $2^3 3^2$ with a quaternion complement of order 2^3. The degree pattern of G is $(1^{(4)}, 2, 8)$.*

From now on, this result will be referred to as the **BCH-theorem**.

In this talk we shall discuss the following three recent papers which generalize the BCH-theorem:

- *"On distinct character degrees"* by Maria Loukaki (2007),

- *"Finite groups whose non-linear irreducible characters of the same degree are Galois conjugate"* by Silvio Dolfi and Manoj Yadav (2016),

and

- *"Finite groups with non-trivial intersections of kernels of all but one irreducible characters"* by Mariagrazia Bianchi, Emanuele Pacifici and myself, which is in its final stages of preparation.

The results of Maria Loukaki.

In her paper

"On distinct character degrees"

published in 2007, Maria Loukaki considered the following problem:

Let $N \neq 1$ be a normal subgroup of the group G and suppose that all irreducible characters of G which **do not contain** N **in their kernel** have distinct degrees (refered to as a group satisfying the (D)-property).

If $N = G'$, then the assumption becomes: **all non-linear** irreducible characters of G have distict degrees, as assumed in the BCH-theorem. Hence the results of Maria Loukaki are a generalization of the BCH-theorem.

The main Theorem A of Loukaki's paper determines **solvable groups** G satisfying the (D)-property with respect to a **minimal normal** subgroup N of order p^n for some prime p. These groups are of three types:

(i) G is a 2-groups of order 2^{2m+1}, with $N = Z(G)$ of order 2 and with a unique faithful irreducible character χ of degree 2^m. In particular, G is a 2-group of central type ($[G : Z(G)] = \chi(1)^2$);

(ii) G is a doubly transitive Frobenius group of order

$$(p^n - 1)p^n$$

with N as its kernel;

(iii) G is neither nilpotent nor Frobenius, but satisfies

$$O_{p'}(G) = 1 = Z(G)$$

and some other property.

Theorem A actually determines all solvable groups G satisfying the (D)-property with respect to some non-necessarily minimal normal subgroup M. Indeed, if N is a minimal normal subgroup of G contained in M, then the irreducible characters of G which do not contain N in their kernel certainly do not contain M in their kernel and hence they have different degrees, as required in Theorem A.

Theorem A of Maria Loukaki was inspired by the 1999-paper of Berkovich, Isaacs and Kazarin:

"Groups with distinct monolithic character degrees",

where in Corollary 4.5 several properties of solvable groups satisfying the hypotheses of Theorem A are derived.

Using Corollary 4.5, the authors provided another proof of the BCH-theorem, but under the additional assumption that G is solvable.

Definitions and comments:

1. A group is said to be a **monolith** if it has exactly one minimal normal subgroup.
2. An irreducible character χ is said to be a **monolithic character** if $G/ker(\chi)$ is a monolith.
3. Every simple group is a monolith and a non-trivial p-group is a monolith if and only if its center is cyclic.

The results of Silvio Dolfi and Manoj Yadav.

In their paper

"Finite groups whose non-linear irreducible characters of the same degree are Galois conjugate"

published in 2016, Silvio Dolfi and Manoj Yadav classified groups G whose **non-linear** irreducible characters which are **not conjugate under the natural Galois action**, have distinct degrees. That means that given two non-linear irreducible characters of G, they are either of different degree or Galois conjugate.

This result is clearly an extension of the BCH-theorem, as well as of the 2013 paper of Dolfi, Navaro and Tiep:

"Finite groups whose same degree characters are Galois conjugate",

where it is assumed that the above assumption is satisfied by **all** non-principal irreducible characters, and not only by the non-linear ones.

Recall that if G is a finite group, n is a multiple of $|G|$ and $\mathfrak{G}_n = Gal(\mathbb{Q}_n|\mathbb{Q})$ is the Galois group of the n-th cyclotomic extension, then $\mathfrak{G}_n$ acts on the set $Irr(G)$ as follows:

for $\alpha \in \mathfrak{G}_n$, $\chi \in Irr(G)$ and $g \in G$ we define

$$\chi^\alpha(g) = \chi(g)^\alpha.$$

If $\chi, \phi \in Irr(G)$ and there exists a Galois automorphism $\alpha \in \mathfrak{G}_n$ such that

$$\chi^\alpha = \phi,$$

then we say that χ and ϕ are **Galois conjugate** (in $\mathfrak{G}_n$).

This is clearly an equivalence relation on $Irr(G)$ and characters in the same class have the same kernel, center, field of values and degree.

In their paper, Silvio Dolfi and Manoj Yadav proved the following theorem:

Theorem A

Every two non-linear irreducible characters of the same degree in a group G are Galois conjugate if and only if G is either abelian or one of the following groups:

(a) G is a p-group (p a prime), $|G'| = p$ and $Z(G)$ is cyclic;

(b) G is a certain Frobenius group with kernel K and complement L, where K is of a prime power order and either elementary abelian or a Suzuki 2-group and L is either cyclic or $L \cong Q_8$;

(c) G is non-solvable and either

$$G \in \{A_8,\ Sz(8),\ J_2,\ J_3,\ L_3(2),\ M_{22},\ Ru,\ Th,\ {}^3D_4(2)\}$$

or

$$G \in \{A_5 \times Sz(8),\ A_5 \times Th,\ L_3(2) \times Sz(8)\}.$$

A Suzuki 2-group H was defined by Graham Higman as a non-abelian 2-group with more then one involution, having a cyclic group of automorphisms which permutes its involutions transitively. Higman showed that

$$\Omega_1(H) = Z(H) = Fr(H) = H'$$

and H is of exponent 4 and class 2.

In their paper, Silvio Dolfi and Manoj Yadav showed that the BCH-theorem follows from their Theorem A.

The results of Mariagrazia Bianchi, Emanuele Pacifici and Marcel Herzog.

It is well known that the intersection of kernels of all irreducible characters of a finite group is trivial. This gives rise to the following question:

Question 1: *Which finite groups have a non-trivial intersection of kernels of all but one irreducible characters?*

We were lead to this problem by considering an apparently more general question:

Question 2: *Which finite groups have two columns in their character table which differ by exactly one entry?*

This problem was suggested to us by our late colleague David Chillag. We shall call groups, satisfying the assumptions of Question 1 or 2, of type 1 or 2, respectively.

Question 2 is apparently more general then Question 1. Indeed, if group G is of type 1, then an intersection of kernels of all but one irreducible characters of G is non-trivial. If $b \neq 1$ belongs to such an intersection, then clearly the column of the character degrees and the column of corresponding to b differ by exactly one entry. The surprising fact is that these two families of finite groups coincide.

Our research concentrated on groups G of type 2, satisfying:

G has two columns in its character table which differ by exactly one entry.

Such groups will be called $CD1$-**groups**. To eliminate trivialities, we shall also assume that the orders of $CD1$-groups G satisfy: $|G| > 2$. From now on, we shall deal mainly with $CD1$-groups.

So let G be a $CD1$-group of order $g > 2$ with k conjugacy classes, and let A and B denote two columns in the character table of G, which differ by exactly one entry. By applying the orthogonality relations, it is easy to see that one of these columns, say A, must be the column of degrees of the irreducible characters of G and the second column B is unique. So we may assume that A and B are the first two columns in the character table and the character table looks as follows:

$$\begin{vmatrix} a_1 & b_1 & c_1 & \dots & z_1 \\ a_2 & b_2 & c_2 & \dots & z_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k-1} & b_{k-1} & c_{k-1} & \dots & z_{k-1} \\ a_k & b_k & c_k & \dots & z_k \end{vmatrix}$$

where $A = (a_1, a_2, \dots, a_k)^t$ (t denotes transposed) is the column of degrees, $B = (b_1, b_2, \dots, b_k)^t$ is the column of b and

$$a_1 = b_1, \ a_2 = b_2, \dots, a_{k-1} = b_{k-1} \quad \text{and} \quad a_k \neq b_k.$$

By the orthogonality relations b_k is a negative integer satisfying

$$-a_k \leq b_k \leq -1$$

and $a_k b_k + (g - a_k^2) = 0$, implying that

$$g = |G| = a_k^2 - b_k a_k \leq 2a_k^2.$$

Since we assume that $g > 2$, the degree a_k is larger than any other degree in $Irr(G)$. In particular, $a_k > 1$.

Moreover, by the orthogonality relations the k-th row of the character table is

$$(a_k, b_k, 0, 0, \ldots, 0)$$

and the corresponding irreducible character χ_k vanishes on all but two conjugacy classes.

Conversely, it is easy to see that if $\chi \in Irr(G)$ vanishes on all but two conjugacy classes, then one of the classes is the identity class and the columns corresponding to these classes in the character table of G differ by exactly one entry. Thus we obtain the following theorem, which summarizes the previous observations.

Theorem 3

Let G be a finite group of order $g > 2$. Then the following properties are equivalent:

1. *The intersection of kernels of all but one irreducible characters of G is non-trivial;*
2. *Two columns in the character table of G differ by exactly one entry;*
3. *The degrees column and another column in the character table of G differ by exactly one entry;*
4. *An irreducible character of G vanishes on all but two conjugacy classes.*

The normal subgroup N of G.

Let G be a group satisfying item (3) in Theorem 3:

The degrees column A and another column B in the character table of G differ by exactly one entry.

Let $Irr(G) = \{\chi_1 = 1_G, \chi_2, \ldots, \chi_k\}$ and suppose that i-th row of the character table of G corresponds to χ_i for each i. Assume also that χ_1 is the principal character 1_G. As shown above, the character table of G (to be denoted from now on by CT(G)) is:

$$\begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ a_2 & a_2 & c_2 & \ldots & z_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k-1} & a_{k-1} & c_{k-1} & \ldots & z_{k-1} \\ a_k & b_k & 0 & \ldots & 0 \end{vmatrix}$$

Clearly χ_k is the only faithful irreducible character of G and B is the only column in the CT(G) which differs from the column A by exactly one entry.

We define now:

$$N = \bigcap_{i=1}^{k-1} \ker \chi_i .$$

Since all the columns of CT(G), other than A and B, vanish on the k-th row, it follows that

$$N = \{1\} \cup \{b^G\}.$$

Hence

b is of order p for some prime p

and

N is a minimal normal subgroup of G.

Thus N is an elementary abelian p-group of order

$$|N| = 1 + |b^G| = p^n \quad \text{for a positive integer } n.$$

Since the CT(G) looks like

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_k & b_k & 0 & \dots & 0 \end{vmatrix}$$

it follows by the orthogonality relations that $a_k + b_k|b^G| = 0$. Hence

$$a_k = -b_k|b^G| = -b_k(p^n - 1).$$

Moreover, since $a_k > 1$, N is contained in the kernel of all linear characters of G and hence

$$N \leq G'.$$

It also follows from the CT(G) that

$$|C_G(x)| = |C_{G/N}(Nx)| \quad \text{for all } x \in G \setminus N.$$

Thus (G, N) is a Camina-pair.

Recall that $g = a_k^2 - b_k a_k$ and $a_k = -b_k|b^G| = -b_k(p^n - 1)$. Hence $a_k - b_k = -b_k p^n$ and

$$g = a_k^2 - b_k a_k = a_k(a_k - b_k) = -a_k b_k p^n = b_k^2(p^n - 1)p^n.$$

In particular, G is of even order.
Since the row of χ_k in the CT(G) is $(a_k, b_k, 0, \dots, 0)$, where $b_k = \chi_k(b)$ and b is of order p, it follows that if Q is a Sylow q-subgroup of G for some prime $q \neq p$, then χ_k vanishes for all $x \in Q \setminus \{1\}$.
Thus the restriction of χ_k to Q is a multiple of the regular character of Q by a positive integer and $|Q|$ divides $\chi_k(1) = a_k$. Consequently g/a_k is a positive power of p and since

$$g/a_k = -b_k p^n,$$

it follows that

$$b_k = -p^t \quad \text{for some non-negative integer } t.$$

Since $a_k = -b_k(p^n - 1)$, it follows that

$$-b_k \text{ is the } p\text{-part of } a_k.$$

Hence the value of a_k determines the value of b_k. In particular, $b_k = -1$ iff $p \nmid a_k$ and $b_k = -a_k$ iff $a_k = p^s$ for some positive integer s.

It follows that

$$a_k = -b_k|b^G| = p^t(p^n - 1)$$

and

$$g = -a_k b_k p^n = p^{n+2t}(p^n - 1) = p^{n+2t}|b^G|.$$

Next we consider $C_G(b)$ and $C_G(N)$. First

$$|C_G(b)| = \frac{g}{|b^G|} = p^{n+2t},$$

so $C_G(b)$ is a Sylow p-subgroup of G. Moreover

$$C_G(N) = \bigcap_{y \in G} C_G(b)^y = O_p(G).$$

Since $O_{p'}(G) \leq C_G(N)$, it follows that $O_{p'}(G) = 1$.

The results of Stephen Gagola.

In his paper

"Characters vanishing on all but two conjugacy classes"

published in 1983, Stephen Gagola investigated the $CD1$-groups from the standpoint of groups having an irreducible character vanishing on all but two conjugacy classes. As stated in Theorem 3, this assumption is equivalent to our assumption that G is a $CD1$-group, meaning that two columns in the character table of G differ by exactly one entry.

In his paper, Stephen Gagola completely determined the structure of

$$G/C_G(N) = G/O_p(G).$$

In particular, he proved that if G is solvable, then a Sylow p-subgroup of $G/O_p(G)$ is abelian. Moreover, if G is solvable, then $G/O_p(G)$ has a normal p-complement, which is isomorphic to the multiplicative group of a near-field. The multiplicative groups of finite near fields are in one-to-one correspondence with the class of doubly transitive Frobenius groups. The finite near-fields have been classified by Hans Zassenhaus in 1936.
In the non-solvable case the result is much more complicated and will be omitted.
The characterization of $O_p(G)$ is an open problem. Stephen Gagola showed that there is no bound on the derived length or the nilpotence class of $O_p(G)$.
Concerning the structure of the group G itself, Stephen Gagola proved that $N = C_G(N)$ if and only if G is a doubly transitive Frobenius group.

Our results.

We are continuing with results of Bianchi, Pacifici and Herzog. In our research we concentrated on the structure of $CD1$-groups themselves, satisfying certain conditions with respect to the entries a_k and b_k. Recall that the character table of a $CD1$-group looks like

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ a_2 & a_2 & c_2 & \dots & z_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k-1} & a_{k-1} & c_{k-1} & \dots & z_{k-1} \\ a_k & b_k & 0 & \dots & 0 \end{vmatrix}$$

and remember that we must still establish the connection between $CD1$-groups and the BCH-theorem.

From now on G denotes a $CD1$-group and the previous notation holds. We continue with a series of results.

Proposition 1

$G' = N$ iff G has only one non-linear irreducible character (by Seitz's Theorem G is either an extra-special 2-group or a doubly transitive Frobenius group with a cyclic complement).

Proof: Suppose that $G' = N$. Then $|G'| = |N| = p^n$ and

$$[G : G'] = g/p^n = p^{n+2t}(p^n - 1)/p^n = p^{2t}(p^n - 1).$$

So the number of linear characters of G is $p^{2t}(p^n - 1)$ and

$$\begin{aligned} a_k^2 + p^{2t}(p^n - 1) &= (p^t(p^n - 1))^2 + p^{2t}(p^n - 1) \\ &= (p^n - 1)p^{2t+n} = g. \end{aligned}$$

So χ_k is the only non-linear irreducible character of G. The opposite direction will be omitted. ∎

Proposition 2

Either $|Z(G)| = 1$ or $Z(G) = \{1, b\}$ and $|Z(G)| = 2$.

Proof: Since the k-th row of the CT(G) is

$$(a_k, b_k, 0, \ldots, 0),$$

it follows that $Z(G) \leq N = \{1\} \cup b^G$. Hence either $b \in Z(G)$ and $|Z(G)| = 2$ or $b \notin Z(G)$ and $|Z(G)| = 1$. ■

Proposition 3

$|Z(G)| = 2$ iff $b_k = -a_k$.

Proof: Recall that $-a_k \leq b_k \leq -1$. Hence $|Z(G)| = 2$ iff $b \in Z(G)$ iff $b_k = -a_k$. ■

The connection between our results and the BCH-theorem will stem from the following two propositions.

Proposition 4

The following statements are equivalent:

1. $G \in CD1$ and $b_k = -1$;
2. G is a doubly transitive Frobenius group.

Proof: Suppose that $G \in CD1$ and $b_k = -1$. Then

$$g = b_k^2|b^G|(|b^G| + 1) = |b_G|p^n = (p^n - 1)p^n$$

and N is a minimal normal subgroup of G of order p^n.
If $x \in N \setminus \{1\}$, then

$$|C_G(x)| = g/|b^G| = p^n = |N|.$$

Hence G is a Frobenius group with N as its kernel and with a complement of order $p^n - 1$. So G is a doubly transitive Frobenius group. We drop the proof of the converse. ■

Proposition 5

The following statements are equivalent:

1. $G \in CD1$ and $b_k = -a_k$;
2. G is a 2-group of central type with $|Z(G)| = 2$.

Comment: Recall that G is a group of central type if there exists $\chi \in Irr(G)$ such that $[G : Z(G)] = \chi(1)^2$.

Proof: Suppose that $G \in CD1$ and $b_k = -a_k$. By Proposition 3, $|Z(G)| = 2$ and $g = a_k^2 - b_k a_k = 2a_k^2$. Hence $[G : Z(G)] = g/2 = a_k^2$ and G is of central type with $|Z(G) = 2$. By a theorem of F. DeMeyer and G.Janusz (1969) concerning groups of central type, if S_q is a Sylow q-subgroup of G for some prime q, then $Z(G) \cap S_q = Z(S_q)$. Since $Z(G) = 2$ it follows that G is a 2-group. Hence G is a 2-group of central type with $|Z(G)| = 2$, as required. The opposite direction requires characters considerations and will be dropped. ■

We are approaching now the BCH-theorem.

Call a $CD1$-group G **extreme** if b_k attains one of the extreme values:

$$-1 \quad \text{or} \quad -a_k.$$

By combining Propositions 4 and 5 we obtain the following Theorem 4, which will be compared with the BCH-theorem.

Theorem 4

A group G is an extreme $CD1$-group iff one of the following holds:

1. *G is a 2-group of central type with $|Z(G)| = 2$;*
2. *G is a doubly transitive Frobenius group.*

Theorem 2 - the BCH-theorem

The non-linear irreducible characters of the non-abelian group G are of distinct degrees if and only if G is of one of the following three types:

1. G is an extra-special 2-group of order 2^{2m+1}. The degree pattern of G is $(1^{(2^m)}, 2^m)$;
2. G is a doubly transitive group of order $(p^n - 1)p^n$ with a cyclic complement.
3. G is the doubly transitive Frobenius group of order $2^3 3^2$, with a quaternion complement of order 2^3.

Now, if item (1) of Theorem 2 holds, then G is an extra-special 2-group of order 2^{2m+1} with degree pattern $(1^{(2^m)}, 2^m)$. Thus $|Z(G)| = 2$ holds by the definition of extra-special 2-groups and

$$[G : Z(G)] = 2^{2m} = (2^m)^2,$$

where 2^m is the degree of an irreducible character of G. Hence G is a 2-group of central type with $|Z(G)| = 2$ and satisfies item (1) of Theorem 4.

Moreover, if either item (2) or item (3) of Theorem 2 holds, then G is a doubly transitive Frobenius group and satisfies item (2) of Theorem 4.

Therefore Theorem 4 is a generalization of the BCH-theorem.

Theorem 4 is a **proper** generalization of the BCH-theorem. Indeed, while the groups satisfying BCH-theorem are solvable, it follows from the Zassenhaus' results that there exist three non-solvable doubly transitive Frobenius groups, which satisfy item (2) of Theorem 4. Denoting by F the Frobenius kernel and by H a Frobenius complement, the non-solvable doubly transitive Frobenius groups of order $(p^2-1)p^2$ for $p = 11, 29, 59$ are:

(i) $G_1 = FH$, where F is elementary abelian of order 11^2 and H is isomorphic to $SL(2,5)$ of order 120.

(ii) $G_2 = FH$, where F is elementary abelian of order 29^2 and H is isomorphic to $H = SL(2,5) \times C_7$ of order 840.

(iii) $G_3 = FH$, where F is elementary abelian of order 59^2 and H is isomorphic to $H = SL(2,5) \times C_{29}$ of order 3480.

The group G_1 is even perfect.
Thus our Theorem 4 properly generalizes the BCH-theorem.

Our other characterization results.

Our next result follows by Proposition 5.

Theorem 5

A group G is a $CD1$-group with $Z(G) \neq 1$ if and only if it is a 2-group of central type with $|Z(G)| = 2$.

Indeed, by Proposition 2, a $CD1$-group with $Z(G) \neq 1$ satisfies $|Z(G)| = 2$ and by Proposition 3 this implies that $b_k = -a_k$. Thus, by Proposition 5, G is a 2-group of central type with $|Z(G)| = 2$. The converse also follows by Proposition 5.

So the open problem remains:

characterize $DC1$-goups with $|Z(G)| = 1$.

Our next result follows by Proposition 4.

Theorem 6

A group G is a $CD1$-group with $p \nmid a_k$ if and only if G is a doubly transitive Frobenius group of order $(p^n - 1)p^n$.

Indeed, we have seen that b_k is equal to the negative of the p-part of a_k. Therefore, if $p \nmid a_k$, then $b_k = -1$ and the result follows by Proposition 4. Moreover, $a_k = p^n - 1$.

Theorem 6 immediately implies the following corollary.

Corollary 7

A group G is a $CD1$-group with $a_k < p$ if and only if $a_k = p - 1$ and G is a doubly transitive Frobenius group of order $(p-1)p$.

On the other hand, Proposition 5 implies the following result. Let s denote a positive integer.

Theorem 8

A group G is a $CD1$-group with $a_k = p^s$ if and only if G is a 2-group of central type with $|Z(G)| = 2$.

Indeed, if $a_k = p^s$, then $b_k = -p^s = -a_k$ and the result follows by Proposition 5.

Theorem 8 immediately implies the following corollary. Let r denote a prime.

Corollary 9

An r-group G is a $CD1$-group if and only if $r = 2$ and G is a 2-group of central type with $|Z(G)| = 2$.

Indeed, if a $CD1$-group G is an r-group, then $p = r$ and $a_k = p^s$ for some positive integer. Thus Theorem 8 applies.

Our final theorem characterizes $CD1$-groups with a_k being a power of a prime.

Theorem 10

Let r denote a prime. Then G is a $CD1$-group with $a_k = r^s$ for some positive integer s if and only if one of the following cases holds:

1. G is a 2-group of central type with $|Z(G)| = 2$ and $|G| = 2^{2s+1}$ for some positive integer s;
2. G is a doubly transitive Frobenius group of order $(2^n - 1)2^n$, where $2^n - 1 = r$ is a Mersenne prime;
3. G is a doubly transitive Frobenius group of order $(3^2 - 1)3^2 = 72$;
4. G is a doubly transitive Frobenius group of order $(p - 1)p$, where $p = 2^n + 1$ is a Fermat prime.

Indeed, if $a_k = r^s$ and $r = p$, then $a_k = p^s$ and (1) holds by Theorem 8. If $r \neq p$, then $a_k = r^s$ implies that $p \nmid a_k$ and by Theorem 6 G is a doubly transitive Frobenius group of order $(p^n - 1)p^n$, with $a_k = p^n - 1$. Hence, by our assumptions,

$$p^n - 1 = r^s$$

and by Lemma 19.3 in Passman's book one of the following must hold:

(i) $s = 1$, $p = 2$ and $r = 2^n - 1$ is a Mersenne prime, yielding (2);

(ii) $s = 3$, $r = 2$, $p = 3$ and $n = 2$, yielding (3); and

(iii) $r = 2$, $n = 1$ and $p = 2^s + 1$ is a Fermat prime, yielding (4).

We conlude this talk with the following interesting corollary of Theorem 10. We shall denote by S_3, D_8 and Q_8 the symmetric group on 3 letters, the dihedral group of order 8 and the quaternion group of order 8, respectively.

Corollary 11

The group G is a $CD1$-group with

$$a_k = r$$

for some prime r if and only either $r = 2$ and G is isomorphic to one of the groups: S_3, D_8 and Q_8, or $r = 2^n - 1$ is a Mersenne prime and G is a doubly transitive Frobenius group of order $(2^n - 1)2^n$.

This corollary is obtained by searching for groups with $a_k = r$ in Theorem 10.

This is the END of my talk.

THANK YOU for your ATTENTION!