The Grothendieck group as a classification tool for algebras

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Example

$A = 2$, $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then $A \sim_E B$ as

$$2 = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix}$$

Some relations among matrices

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Definition

The equivalence relation generated by $\sim_E$ is called strongly shift equivalent, denoted by $\sim_S$, i.e., $A \sim_S B$ if

$$A = A_0 \sim_E A_1 \sim_E A_2 \sim_E \cdots \sim_E A_n = B.$$

Example

Let $A = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 6 \\ 1 & 1 \end{pmatrix}$. Question: $A \sim_S B$?

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$$A_1 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

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$$A_1 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} =$$

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$$\begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{pmatrix} = A_2$$

Example

Let $A = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 6 \\ 1 & 1 \end{pmatrix}$. Question: $A \sim_S B$?

$$\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \sim_E A_1 \sim_E A_2$$

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} = A_1$$

$$A_1 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{pmatrix} = A_2$$

$$A_2 = \begin{pmatrix} 1 & 2 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{pmatrix}$$

Example

Let $A = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 6 \\ 1 & 1 \end{pmatrix}$. Question: $A \sim_S B$?

$$\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \sim_E A_1 \sim_E A_2$$

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} = A_1$$

$$A_1 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{pmatrix} = A_2$$

$$A_2 = \begin{pmatrix} 1 & 2 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Example

$$\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \sim_E A_1 \sim_E A_2$$

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

Example

$$\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \sim_E A_1 \sim_E A_2 \sim_E A_3$$

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix} = A_3$$

Example

$$\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \sim_E A_1 \sim_E A_2 \sim_E A_3$$

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix} = A_3$$

$$A_3 = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$$

Example

$$\begin{pmatrix}1 & 3\\ 2 & 1\end{pmatrix} \sim_E A_1 \sim_E A_2 \sim_E A_3$$

$$\begin{pmatrix}0&1&1&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix}\begin{pmatrix}2&0&0&1\\0&2&0&1\\1&0&1&0\\1&1&0&1\end{pmatrix} = \begin{pmatrix}1&2&1&1\\1&0&1&0\\1&1&0&1\\2&0&0&1\end{pmatrix} = A_3$$

$$A_3 = \begin{pmatrix}1&2&1&1\\1&0&1&0\\1&1&0&1\\2&0&0&1\end{pmatrix} = \begin{pmatrix}0&1&1&0\\0&0&0&1\\0&1&0&0\\1&0&0&0\end{pmatrix}\begin{pmatrix}2&0&0&1\\1&1&0&1\\0&1&1&0\\1&0&1&0\end{pmatrix}$$

Example

$$\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \sim_E A_1 \sim_E A_2 \sim_E A_3$$

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix} = A_3$$

$$A_3 = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Example

$$\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \sim_E A_1 \sim_E A_2 \sim_E A_3 \sim_E A_4$$

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix} = A_3$$

$$A_3 = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix} = A_4$$

Example

$$\begin{pmatrix}1 & 3\\ 2 & 1\end{pmatrix} \sim_E A_1 \sim_E A_2 \sim_E A_3 \sim_E A_4 \sim_E A_5$$

$$\begin{pmatrix}0&1&1&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix}\begin{pmatrix}2&0&0&1\\0&2&0&1\\1&0&1&0\\1&1&0&1\end{pmatrix} = \begin{pmatrix}1&2&1&1\\1&0&1&0\\1&1&0&1\\2&0&0&1\end{pmatrix} = A_3$$

$$A_3 = \begin{pmatrix}1&2&1&1\\1&0&1&0\\1&1&0&1\\2&0&0&1\end{pmatrix} = \begin{pmatrix}0&1&1&0\\0&0&0&1\\0&1&0&0\\1&0&0&0\end{pmatrix}\begin{pmatrix}2&0&0&1\\1&1&0&1\\0&1&1&0\\1&0&1&0\end{pmatrix}$$

$$\begin{pmatrix}2&0&0&1\\1&1&0&1\\0&1&1&0\\1&0&1&0\end{pmatrix}\begin{pmatrix}0&1&1&0\\0&0&0&1\\0&1&0&0\\1&0&0&0\end{pmatrix} = \begin{pmatrix}1&2&2&0\\1&1&1&1\\0&1&0&1\\0&2&1&0\end{pmatrix} = A_4$$

$$A_4 = \begin{pmatrix}1&2&2&0\\1&1&1&1\\0&1&0&1\\0&2&1&0\end{pmatrix} = \begin{pmatrix}0&1&1&1\\1&0&1&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}0&1&0&1\\0&2&1&0\\0&0&1&0\\1&0&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&1&0&1\\0&2&1&0\\0&0&1&0\\1&0&0&0\end{pmatrix}\begin{pmatrix}0&1&1&1\\1&0&1&1\\1&0&0&0\\0&1&0&0\end{pmatrix} = \begin{pmatrix}1&1&1&1\\3&0&2&2\\1&0&0&0\\0&1&1&1\end{pmatrix} = A_5$$

Example

$$\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \sim_E A_1 \sim_E A_2 \sim_E A_3 \sim_E A_4 \sim_E A_5$$

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix} = A_3$$

$$A_3 = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix} = A_4$$

$$A_4 = \begin{pmatrix} 1 & 2 & 2 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 0 & 2 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} = A_5$$

$$A_5 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 0 & 2 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 3 & 0 & 2 & 2 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Example

$$\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \sim_E A_1 \sim_E A_2 \sim_E A_3 \sim_E A_4 \sim_E A_5$$

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 3 & 0 & 2 & 2 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} =$$

Example

$$\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \sim_E A_1 \sim_E A_2 \sim_E A_3 \sim_E A_4 \sim_E A_5 \sim_E A_6$$

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 3 & 0 & 2 & 2 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 5 & 0 & 5 \\ 1 & 0 & 1 \end{pmatrix} = A_6$$

$$A_6 = \begin{pmatrix} 1 & 1 & 1 \\ 5 & 0 & 5 \\ 1 & 0 & 1 \end{pmatrix}$$

Example

$$\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \sim_E A_1 \sim_E A_2 \sim_E A_3 \sim_E A_4 \sim_E A_5 \sim_E A_6$$

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 3 & 0 & 2 & 2 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 5 & 0 & 5 \\ 1 & 0 & 1 \end{pmatrix} = A_6$$

$$A_6 = \begin{pmatrix} 1 & 1 & 1 \\ 5 & 0 & 5 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 5 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Example

$$\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \sim_E A_1 \sim_E A_2 \sim_E A_3 \sim_E A_4 \sim_E A_5 \sim_E A_6$$

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 3 & 0 & 2 & 2 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 5 & 0 & 5 \\ 1 & 0 & 1 \end{pmatrix} = A_6$$

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Some relations among matrices

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Let $A_k = \begin{pmatrix} 1 & k \\ k-1 & 1 \end{pmatrix}$ *and* $B_k = \begin{pmatrix} 1 & k(k-1) \\ 1 & 1 \end{pmatrix}$.

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Is

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$A \sim B$, *shift equivalent* if $\exists\, R, S$: non-negative integer matrices such that for $i > 0$,

$$A^i = RS$$
$$B^i = SR$$
$$AR = RB, \quad SA = BS.$$

R. Williams, Classification of shift of finite type, Ann. of Math. 1973.

Some relations among matrices

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Theorem (R. Williams)

1. *$A \sim_E B$ implies $A \sim B$.*
2. *$A \sim_S B$ if and only if $A \sim B$.*

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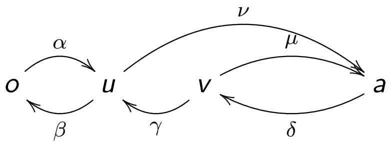
Counterexample (Kim, Rousch, William's conjecture is false, Ann. of Math. 1992)

$A \sim B$ *does not imply* $A \sim_S B$.

$E:$ 

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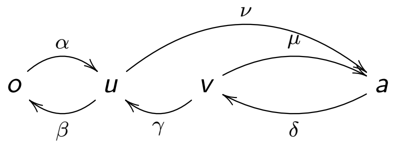

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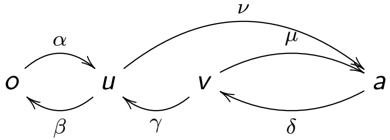

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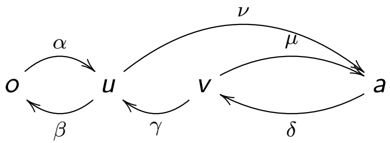

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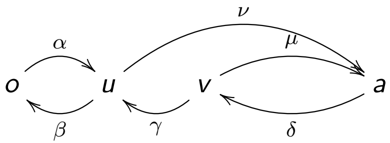

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Pictorial approach

A non-negative square matrix $\Longleftrightarrow$ finite directed graph

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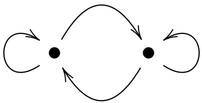

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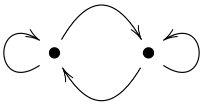

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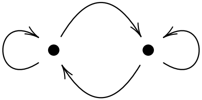

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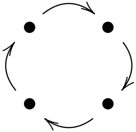

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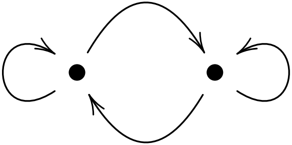

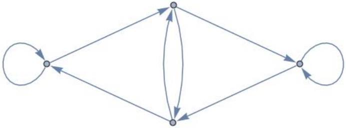

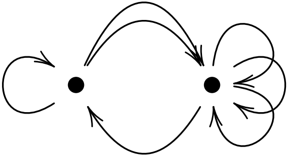

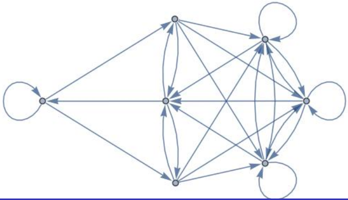

Matrices $\Longleftrightarrow$ Graphics

Theorem (Williams)

$A \sim_S B$ if and only if there is a sequence of insplit and outsplit from A to B.

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Invariants: Let A be a $n \times n$ non-negative square matrix.

$$\begin{array}{ccccccc}
\mathbb{Z}^n & \xrightarrow{A} & \mathbb{Z}^n & \xrightarrow{A} & \mathbb{Z}^n & \xrightarrow{A} & \cdots \\
\downarrow{\scriptstyle A} & & \downarrow{\scriptstyle A} & & \downarrow{\scriptstyle A} & & \\
\mathbb{Z}^n & \xrightarrow{A} & \mathbb{Z}^A & \xrightarrow{A} & \mathbb{Z}^n & \xrightarrow{A} & \cdots
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\qquad\qquad \vdots\,\delta_A$$

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Theorem (W. Krieger, Dimension function and topological Markov chains, Invent. Math, 1980)

$A \sim B$ if and only if $(\Delta_A, \Delta_A^+, \delta_A) \cong (\Delta_B, \Delta_B^+, \delta_B)$.

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Grothendieck group K_0

Let A be a ring with identity.

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$K_0(A)$ is a pre-ordered abelian group with an order unit $[A]$.

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Example

$$K \longrightarrow \mathbb{M}_2(K) \longrightarrow \mathbb{M}_4(K) \longrightarrow \dots$$

$$a \longmapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

$$K \oplus K \longrightarrow \mathbb{M}_2(K) \oplus K \longrightarrow \mathbb{M}_3(K) \oplus \mathbb{M}_2(K) \longrightarrow \cdots$$

$$(a, b) \longmapsto (\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, a)$$

Theorem (Bratteli, Elliott, Goodearl)

Let R and S be ultramatricial K-algebra. Then $R \cong S$ as K-algebra if and only if

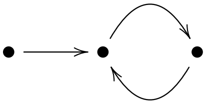
$$\big(K_0(R), K_0(R)_+, [R]\big) \cong \big(K_0(S), K_0(S)_+, [S]\big).$$

Classification of LPAs via K-groups

F

$$\mathcal{L}(F) \cong \mathbb{M}_3(K)$$

E



$$\mathcal{L}(E) \cong \mathbb{M}_3(K[x, x^{-1}])$$

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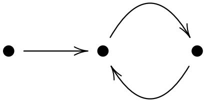
E  $\mathcal{L}(E) \cong \mathbb{M}_3(K[x, x^{-1}])$

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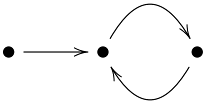
E  $\mathcal{L}(E) \cong \mathbb{M}_3(K[x, x^{-1}])$

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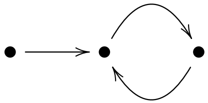

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$$\Big(K_0(\mathcal{L}(E)), K_0(\mathcal{L}(E))_+, [\mathcal{L}(E)]\Big) \cong (\mathbb{Z}, \mathbb{N}, 3)$$

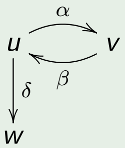
But

$$\mathbb{M}_3(K) \not\cong \mathbb{M}_3(K[x, x^{-1}]).$$

So K_0 doesn't seem to classify all types of LPAs.

G. Abrams, G. Aranda Pino, The Leavitt path algebra of a graph, J. Algebra (2005).

Example (Double of a graph)



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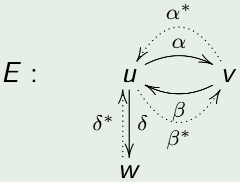

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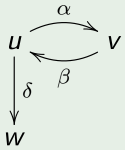

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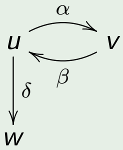

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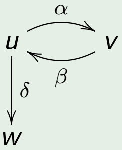

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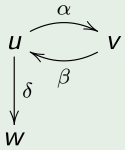

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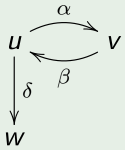

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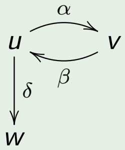

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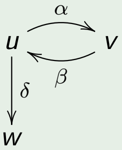
$\alpha^*\alpha = v$,
$\alpha\alpha^* + \delta\delta^*$

Leavitt path algebras

Example (Relations in Leavitt path algebra)

E :



$\alpha\beta\alpha \in \mathcal{L}(E)$,
$\alpha^*\alpha = v$,
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Definition

For a graph E, let $\mathcal{L}(E)$ be the algebra generated by the sets $\{v \mid v \in E^0\}$, $\{\alpha \mid \alpha \in E^1\}$ and $\{\alpha^* \mid \alpha \in E^1\}$ subject to the relations

1. $uv = \delta_{u,v} u$ for every $u, v \in E^0$.
2. $s(\alpha)\alpha = \alpha r(\alpha) = \alpha$ and $r(\alpha)\alpha^* = \alpha^* s(\alpha) = \alpha^*$ for all $\alpha \in E^1$.

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3. $\alpha^*\alpha' = \delta_{\alpha\alpha'} r(\alpha)$, for all $\alpha, \alpha' \in E^1$.
4. $\sum_{\{\alpha \in E^1, s(\alpha)=v\}} \alpha\alpha^* = v$ for every $v \in E^0$ for which $s^{-1}(v)$ is non-empty.

Theorem (Abrams, Aranda Pino, 2005)

$\mathcal{L}_K(E)$ is simple if and only if

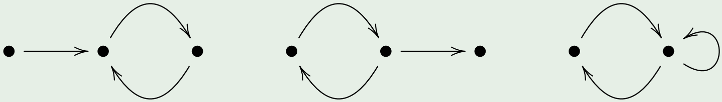
Leavitt path algebras: Algebras we can see

Theorem (Abrams, Aranda Pino, 2005)

$\mathcal{L}_K(E)$ *is* simple *if and only if*

1. *Every vertex connects to every cycle and to every sink in* E*, and*
2. *Every cycle in* E *has an exit.*

Example



LPA arises in a variety of different context...

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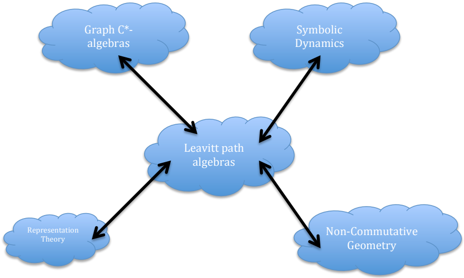

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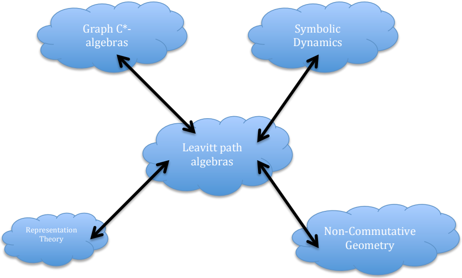

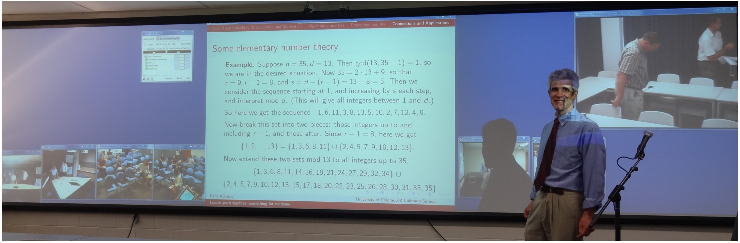

Gene Abrams at UWS, Feb 2013.

Towards grading of LPAs

- K-theory does not seem to capture enough information.

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Example (Acyclic graphs)

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Example (Acyclic graphs)

$$F: \begin{array}{ccc} & & \bullet \\ & & \downarrow \\ \bullet & \longrightarrow & u \end{array} \qquad\qquad E: \bullet \longrightarrow \bullet \longrightarrow u$$

Then

$$\mathcal{L}(F) \cong \mathcal{L}(E) \cong \mathbb{M}_3(K).$$

Towards grading of LPAs

- Taking grading into account...

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Example (Acyclic graphs with the grading)

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Example (Acyclic graphs with the grading)

Then

$$\mathcal{L}(F) \cong_{\text{gr}} \mathbb{M}_3(K)(0,1,1) \qquad \mathcal{L}(E) \cong_{\text{gr}} \mathbb{M}_3(K)(0,1,2).$$

For a Γ-graded ring A with identity and a graded finitely generated projective (right) A-module P, let $[P]$ denote the class of graded A-modules graded isomorphic to P. Then the monoid

Graded Grothendieck group K_0^{gr}

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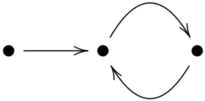
The group $\mathcal{V}^{\mathrm{gr}}(A)^{+}$ is called the graded Grothendieck group and is denoted by $K_0^{\mathrm{gr}}(A)$, which is a $\mathbb{Z}[\Gamma]$-module.

Graded versus non-graded K-theory

F  $\mathcal{L}(F) \cong \mathbb{M}_3(K)$

E  $\mathcal{L}(E) \cong \mathbb{M}_3(K[x, x^{-1}])$

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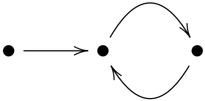

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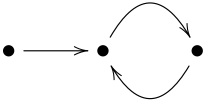

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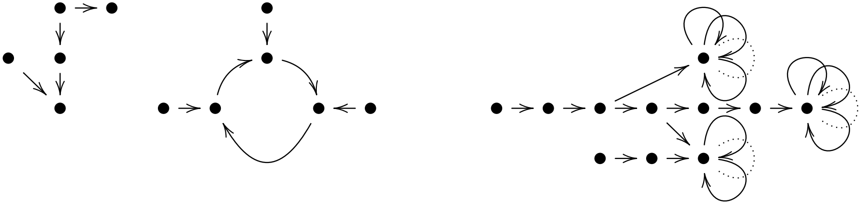
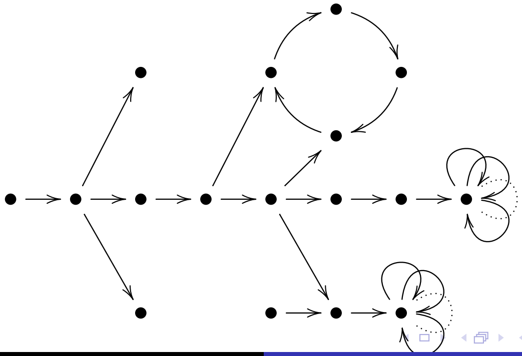
Polycephaly graphs





Conj: Graded K-theory classifies *all* LPAs

Theorem

Let E and F be polycephaly graphs. Then $\mathcal{L}(E) \cong_{\text{gr}} \mathcal{L}(F)$ if and only if there is a $\mathbb{Z}[x, x^{-1}]$-module isomorphism

$$\big(K_0^{\text{gr}}(\mathcal{L}(E)), [\mathcal{L}(E)]\big) \cong \big(K_0^{\text{gr}}(\mathcal{L}(F)), [\mathcal{L}(F)]\big).$$

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Let E and F be finite graphs. Then $\mathcal{L}(E) \cong_{\mathrm{gr}} \mathcal{L}(F)$ if and only if there is an order $\mathbb{Z}[x, x^{-1}]$-module isomorphism

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Theorem (Ara, Pardo, 2014, *J. K-theory*)

A weak version of the conjecture is valid for finite graphs with no sinks and sources.

E and F finite graphs, A_E and A_F the adjacency matrices and X_E and X_F are shift of finite types.

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