Cycles and Bipartite Divisor Graph on Irreducible Character Degrees

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Outline

- Bipartite divisor graph for the set of conjugacy class sizes
- Bipartite divisor graph for the set of irreducible character degrees
- Groups where $\mathrm{B}(G)$ is a cycle

Bipartite divisor graph for conjugacy class sizes

Given a finite group G, there are several sets of invariants that convey nontrivial information about the structure of G.
Some classical examples include the set consisting of the orders of the elements of G, or the set of conjugacy class sizes of G, or the set of degrees of the irreducible complex characters of G.
For every set X as above, it is natural to ask to what extent the group structure of G is reflected and influenced by X,
and a useful tool in this kind of investigation is the so-called *bipartite divisor graph* $\mathrm{B}(X)$ of X.

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and a useful tool in this kind of investigation is the so-called *bipartite divisor graph* $\mathrm{B}(X)$ of X.

Definition

Bipartite Divisor Graph The bipartition of the vertex set of $\mathrm{B}(X)$ *consists of* $X \setminus \{1\}$ *and of the set of prime numbers dividing* x*, for some* $x \in X$*, and the edge set of* $\mathrm{B}(X)$ *consists of the pairs* $\{p, x\}$ *with* $\gcd(p, x) \neq 1$*.*

review

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Bubboloni, Dolfi, Iranmanesh and Praeger, 2009

Theorem

For a finite group G, $\mathrm{diam}(B(G)) \leqslant 6$*. Moreover,* $\mathrm{diam}(B(G)) = 6$ *if and only if one of the following occurs:*

(i) $G = (A \rtimes B) \times C$, *where* A, B *and* C *are groups of pairwise coprime order,* A *and* B *are abelian,* C *is nonabelian and* $AB/Z(AB)$ *is a Frobenius group.*

(ii) $G = (A \times C) \rtimes B$, *where* A, B *and* C *are abelian groups,* A *and* C *are normal in* G, $\gcd(|A||C|, |B|) = 1$ *and* $AB/(B \cap Z(G)$ *is a Frobenius group.*

Moreover, if $\mathrm{diam}(B(G)) = 6$, *then the graph* $B(G)$ *is connected.*

Theorem

Let G be a finite group. Assume that $B(G)$ is a path. Then one of the following occurs:

- (i) G *is one of the groups of types* A, B *and* C.
- (ii) G *is a* p*-group for some prime* p.
- (iii) $G = KL$, *with* $K \trianglelefteq G$, $\gcd(|K|, |L|) = 1$ *and one of the following cases occurs:*
  - (a) *both* K *and* L *are abelian,* $Z(G) \leqslant L$ *and* $G/Z(G)$ *is a Frobenius group.*
  - (b) K *is abelian,* L *is a nonabelian* p*-group for some prime* p, $O_p(G)$ *is an abelian subgroup of index* p *in* L *and* $G/O_p(G)$ *is a Frobenius group.*
  - (c) L *is abelian,* K *is a* p*-group of conjugate rank one,* $Z(K) = Z(G) \cap K$ *and* $G/Z(G)$ *is a Frobenius group.*
- (iv) G *is group with* $cs(G) = \{1, p^{\alpha}, q^{\beta}, p^{\gamma}q^{\theta}\}$.
- (v) G *is a direct product of a* p*-group and an abelian* p'*-group.*

Furthermore, G *is solvable.*

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Let G be a finite group. Then $B(G)$ is a cycle if and only if $B(G)$ is a cycle of length six and $G \simeq A \times SL_2(q)$, where A is an abelian group and $q \in \{4, 8\}$.

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Is there any finite group G such that $B(G)$ is isomorphic to a complete bipartite graph $K_{m,n}$, for some positive integers $m, n \geqslant 2$?

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Casolo's Conjecture If G is a finite group with $B(G) = K_{m,n}$ (where n is the number of non-identity conjugacy class sizes of G), then $n \geqslant 2^m$.

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Hafezieh; Bipartite divisor graphs for the set of irreducible character degrees

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Example

$\mathrm{cd}(\mathrm{G}) = \{1, 3, 5, 3 \times 5, 7 \times 31 \times 151, 2^7 \times 7 \times 31 \times 151, 2^{12} \times 31 \times 151, 2^{12} \times 3 \times 31 \times 151, 2^{12} \times 7 \times 31 \times 151, 2^{13} \times 7 \times 31 \times 151, 2^{15} \times 3 \times 31 \times 151\}$

It is easy to see that $\mathrm{diam}(\mathrm{B}(\mathrm{G})) = 7$.

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Hafezieh; Groups whose bipartite divisor graphs are cycles

Theorem

Let G be a finite group whose $B(G)$ is a cycle of length n. Then we have the following properties:

(i) $n \in \{4, 6\}$.

(ii) G *is solvable and* $\mathrm{dl}(G) \leqslant |\mathrm{cd}(G)| \leqslant 4$.

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Examples of group whose $B(G)$ is a cycle

Example

For every pair of odd primes like p and q such that p is congruent to 1 modula 3 and q is a prime divisor of $p+1$, there exists a solvable group G such that $cd(G) = \{1, 3q, p^2q, 3p^3\}$. This gives an example of a solvable group G whose bipartite divisor graph related to the set of character degrees, is a cycle of length 6.

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Example

There are exactly 66 *groups of order* 588. *Among these groups, there are exactly two nonabelian groups whose bipartite divisor graphs are cycles of length four. These groups have* $\{1, 6, 12\}$ *as their irreducible character degrees set.*

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Theorem

Let G be a finite group. Assume that $B(G)$ is a cycle of length 4. There exists a normal abelian subgroup N of G such that
$cd(G) = \{[G : I_G(\lambda)] : \lambda \in Irr(N)\}$.

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