Finite Groups with 6 Automorphism Orbits

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In this talk all considered groups are assumed to be finite.

Let G be a group. An "automorphism orbit" of G is a subset of G of this form:

$$\{g^\alpha \ : \ \alpha \in \mathrm{Aut}(G)\}$$

where g is a fixed element of G. In other words, an automorphism orbit of G is an orbit of the natural action of $\mathrm{Aut}(G)$ on G.

Let $\omega(G)$ denote the **number of automorphism orbits of G**.

Obviously $\omega(G) = 1$ if and only if $G = \{1\}$. It is well-known that $\omega(G) = 2$ if and only if G is elementary abelian.

Theorem (Landau)

There are only finitely many finite groups with a given number of conjugacy classes.

This does not happen with automorphism orbits, for instance all elementary abelian groups have exactly two automorphism orbits. We proved that

Theorem

There are only finitely many finite groups without nontrivial abelian normal subgroups with a given number of automorphism orbits.

Proof.

Let N be the socle of G, $C = C_G(N) \trianglelefteq G$. If $C \neq \{1\}$ there is some minimal normal subgroup M of G contained in C. M centralizes N, in particular M centralizes M, contradicting the hypothesis. This proves that $C = \{1\}$, so G embeds in $\mathrm{Aut}(N)$. So it is enough to bound $|N|$. Since N is characteristic, $\omega(N) \leq \omega(G)$, and N is a direct product of nonabelian simple groups. The result follows from Kohl's work on the number of automorphism orbits of nonabelian simple groups. □

It is indeed not possible to improve this result.

Consider a field F of characteristic 2, m a positive integer, $V := M(2 \times m, F)$ and $G := V \rtimes SL(2, F)$ with the natural action of $SL(2, F)$ on V given by left multiplication.

We proved that if $m \geq 2$ then

$$\omega(G) = 3 + \omega(SL(2, F)).$$

In particular, $\omega(G)$ does not depend on m.

For example if $m \geq 2$ and $F = \mathbb{F}_4$ we get $\omega(G) = 7$.

If $m = 1$, $F = \mathbb{F}_4$ then $G = ASL(2, 4)$ and $\omega(G) = 6$.

Laffey and MacHale classified the finite groups G with $\omega(G) = 3$ and showed that A_5 (which has $\omega(A_5) = 4$) is the only nonsolvable finite group G such that $\omega(G) \leq 4$.

Markus Stroppel showed that the only finite nonabelian simple groups S with $\omega(S) \leq 5$ are the groups $PSL(n, q)$ with $q \in \{4, 7, 8, 9\}$. He also proposed the following problem.

M. STROPPEL'S PROBLEM. Classify the finite nonsolvable groups G with $\omega(G) \leq 6$.

We solved this problem. Our result is the following.

THEOREM

If G is a finite nonsolvable group with $\omega(G) \leq 6$ then G is isomorphic to one of $PSL(2, q)$ with $q \in \{4, 7, 8, 9\}$, $PSL(3, 4)$ or $ASL(2, 4)$.

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Here $ASL(2,4)$ denotes the semidirect product $\mathbb{F}_4^2 \rtimes SL(2,4)$ with the natural action of $SL(2,4)$ on $\mathbb{F}_4^2$. Call $V := \mathbb{F}_4^2$ and $H := SL(2,4)$. The automorphism orbits of $G = ASL(2,4) = V \rtimes H$ are the following:

$$\{1\},\ \{g \in G\ :\ o(g) = 3\},\ \{g \in G\ :\ o(g) = 5\},$$

$$\{g \in G\ :\ o(g) = 4\},\ \{g \in V\ :\ o(g) = 2\},\ \{g \in G - V\ :\ o(g) = 2\}.$$

Thus $\omega(ASL(2,4)) = 6$.

Observe that V is characteristic in G and that $G/V \cong SL(2,4) \cong A_5$.

To see how this is proved let us first state an useful property of the function ω.

LEMMA

Let G be a group and let N be a characteristic subgroup of G. Then $\omega(G) \geq \omega(N) + \omega(G/N) - 1$.

The idea is that the nontrivial automorphism orbit representatives of G/N induce automorphism orbits of G disjoint from N.

Another useful tool we used extensively is Zhang's classification of the finite groups G in which two elements of the same order lie in the same automorphism orbit.

Now assume $\omega(G) = 6$, and let N be a nontrivial characteristic subgroup of G. It is not hard to show that $Z(G) = \{1\}$. The hard part is to bound $|N|$ when N is (elementary) abelian.

PROPOSITION

Suppose $x_1, \ldots, x_m \in G$ are automorphism-conjugated elements such that $\langle x_1, \ldots, x_m \rangle N = G$ and $\bigcup_{i=1}^m C_N(x_i) = N$. Then $|N| \leq m^m$.

PROOF.

Setting $k := |C_N(x_i)|$ for all i, since $\bigcup_{i=1}^m C_N(x_i) = N$ we have

$$|N| \leq m(k-1)+1 \qquad \Rightarrow \qquad mk \geq |N|.$$

Since $\langle x_1 N, \ldots, x_m N \rangle = G/N$ and $Z(G) = \{1\}$, $\bigcap_{i=1}^m C_N(x_i) = \{1\}$ so

$$\begin{aligned} |N| &= |N : \bigcap_{i=1}^m C_N(x_i)| \leq \prod_{i=1}^m |N : C_N(x_i)| = \\ &= (|N|/k)^m \leq (|N|/(|N|/m))^m = m^m. \end{aligned}$$

□