Beauville structures in finite p-groups

Gustavo A. Fernández-Alcober

University of the Basque Country, Bilbao, Spain

(joint work with Şükran Gül)

Dedicated to the memory of Guido Zappa, Mario Curzio, and Wolfgang Kappe

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Definition

A Beauville surface of unmixed type is a compact complex surface which is the quotient of a product $C_1 \times C_2$ of two algebraic curves C_1 and C_2 of genera at least 2 by the action of a finite group G acting freely by holomorphic transformations, in such a way that:

(i) $C_i/G \cong \mathbb{P}_1(\mathbb{C})$ for $i = 1, 2$.

(ii) The covering map $C_i \to C_i/G$ is ramified over three points for $i = 1, 2$.

The group G is then called a Beauville group of unmixed type.

Definition

Let G be a group. For every $x, y \in G$, we define

$$\Sigma(x,y) = \bigcup_{g \in G} \left(\langle x \rangle^g \cup \langle y \rangle^g \cup \langle xy \rangle^g \right).$$

Theorem

A finite group G is a Beauville group if and only if:

(i) *G is a 2-generator group.*

(ii) *G has two sets of generators, $\{x_1, y_1\}$ and $\{x_2, y_2\}$, such that*

$$\Sigma(x_1, y_1) \cap \Sigma(x_2, y_2) = 1.$$

We say that $\{x_1, y_1\}$ and $\{x_2, y_2\}$ is a Beauville structure for G.

Main Problem

What finite groups are Beauville groups?

In the sequel, all groups will be finite and p will always be a prime.

Theorem (Catanese, 2000)

An abelian group is a Beauville group if and only if it is isomorphic to $C_n \times C_n$*, where* $\gcd(n, 6) = 1$.

Corollary

There are no abelian Beauville 2-groups or 3-groups.

Corollary

Let $p \geq 5$*, and let* G *be an abelian 2-generator* p*-group. If the exponent of* G *is* p^e*, then:*

$$G \text{ is a Beauville group} \iff |G^{p^{e-1}}| = p^2.$$

Theorem (Bauer, Catanese, Grunewald, 2005; Fuertes, González-Diez, 2009)

The alternating groups A_n, for $n \geq 6$, and the symmetric groups S_n, for $n \geq 5$, are Beauville groups.

Theorem (Bauer, Catanese, Grunewald, 2005)

The groups $SL(2,p)$, $PSL(2,p)$ for $p \neq 2,3,5$, are Beauville groups.

Theorem (Guralnick, Malle, 2012)

Any non-abelian finite simple group other than A_5 is a Beauville group.

Beauville p-groups: previously known results

If G is of exponent p, it is easy to see that:

$$G \text{ is a Beauville group} \iff p \geq 5 \text{ and } |G| \geq p^2.$$

Barker, Boston, and Fairbairn (2012)

- All Beauville p-groups of order at most p^4 are determined.
- Estimates are given for the number of Beauville groups of orders p^5 and p^6.
- Not all cases are settled. For example, it is conjectured that the following groups of order p^5 are Beauville groups for $p \geq 5$:

$$\langle a, b, c \mid a^{p^2} = b^{p^2} = c^p = [b, c] = 1,\ [a, b] = c,\ [a, c] = b^{rp} \rangle$$

(r not divisible by p)

Beauville p-groups: previously known results

Barker, Boston, and Fairbairn (2012)

- There are non-abelian Beauville p-groups of order p^n for every $p \geq 5$ and $n \geq 3$. If $n = 2m$ is even, we have the group

$$\langle a, b \mid a^{p^m} = b^{p^m} = 1,\ [a, b] = a^p \rangle,$$

  and if $n = 2m + 1$ is odd,

$$\langle a, b, c \mid a^{p^m} = b^{p^m} = c^p = 1,\ [a, b] = c,\ [a, c] = [b, c] = 1 \rangle.$$

- The smallest non-abelian Beauville p-groups for $p = 2$ and $p = 3$ are of order 2^7 and 3^5.

Some of these results rely on computations with computer algebra systems such as GAP or MAGMA.

Beauville p-groups: previously known results

Theorem (Stix, Vdovina, 2015)

A split metacyclic p-group G is a Beauville group if and only if $p \geq 5$ and G is a semidirect product of two cyclic groups of the same order.

Theorem (Barker, Boston, Peyerimhoff, and Vdovina, 2015)

There are infinitely many Beauville 2-groups.

Theorem (González-Diez, Jaikin-Zapirain, 2015)

Every 2-generator group G can be expressed as a quotient of a Beauville group $\widetilde{G}$ such that $\pi(\widetilde{G}) = \pi(G)$.

Corollary

There are infinitely many Beauville 3-groups.

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Definition

Let G be a p-group of exponent p^e. We say that G satisfies condition SA (SA from semi-abelian) if

$$x^{p^{e-1}} = y^{p^{e-1}} \iff (xy^{-1})^{p^{e-1}} = 1.$$

Theorem (F-A, Gül)

Let G be a p-group of exponent p^e satisfying condition SA. Then:

$$G \text{ is a Beauville group} \iff p \geq 5 \text{ and } |G^{p^{e-1}}| \geq p^2.$$

Furthermore, any lift to G of a Beauville structure of $G/\Phi(G)$ is a Beauville structure of G.

Almost all families of p-groups which are known to have a good behaviour with respect to powers satisfy condition SA:

- Obviously, abelian groups and groups of exponent p.
- Regular p-groups.
- Thus in particular groups of order $\leq p^p$ and groups with $\gamma_{p-1}(G)$ cyclic.
- Thus also metacyclic p-groups for odd p, and we get as a corollary the Stix-Vdovina criterion in that case.
- Powerful p-groups, i.e. groups in which $G' \leq G^p$ if p is odd, or $G' \leq G^4$ if $p = 2$.
- The so-called p-central p-groups, i.e. groups in which $\Omega_1(G) \leq Z(G)$ if p is odd, or $\Omega_2(G) \leq Z(G)$ if $p = 2$.

If G is a p-group, recall that $\Omega_i(G) = \langle g \in G \mid g^{p^i} = 1 \rangle$.

Generalised p-central groups and potent p-groups

Definition

Let G be a p-group. Then:

(i) G is generalised p-central if $\Omega_1(G) \leq Z_{p-2}(G)$ for odd p, and if $\Omega_2(G) \leq Z(G)$ if $p = 2$.

(ii) G is potent if $\gamma_{p-1}(G) \leq G^p$ for odd p, and if $G' \leq G^4$ for $p = 2$.

Potent p-groups have been studied by González-Sánchez and Jaikin-Zapirain, and generalised p-central p-groups by González-Sánchez and Weigel.

Theorem (F-A, Gül)

A generalised p-central p-group satisfies condition SA. However, not all potent p-groups satisfy condition SA.

Thus our criterion for groups with condition SA does not apply to potent p-groups. But we can prove that it is valid all the same.

Theorem (F-A, Gül)

Let G be a potent p-group of exponent p^e. Then:

$$G \text{ is a Beauville group} \iff p \geq 5 \text{ and } |G^{p^{e-1}}| \geq p^2.$$

Furthermore, any lift to G of a Beauville structure of $G/\Phi(G)$ is a Beauville structure of G.

Our criterion makes it easy to check whether a group given by a presentation is Beauville, provided that it satisfies condition SA or that it is potent.

Example

Consider, for $p \geq 5$, the group G of order p^5 whose behaviour was unknown in the paper by Baker, Boston, and Fairbairn:

$$\langle a, b, c \mid a^{p^2} = b^{p^2} = c^p = [b, c] = 1,\ [a, b] = c,\ [a, c] = b^{rp} \rangle$$

(r not divisible by p)

We have $\exp G = p^2$. Since $\langle a^p \rangle \neq \langle b^p \rangle$ are non-trivial, then $|G^p| \geq p^2$ and we conclude that G is a Beauville group.

More generally, let $G = \langle a, b \rangle$ be a p-group either satisfying SA or potent, with $p \geq 5$. If $\exp G = p^e$, then:

$$G \text{ is a Beauville group} \iff \langle a^{p^{e-1}} \rangle \neq \langle b^{p^{e-1}} \rangle \text{ are non-trivial.}$$

If $|G| \leq p^p$, we can apply the Lazard Correspondence.

Then G can be given the structure of a Lie ring, and $ng = g^n$ for all $g \in G$ and $n \in \mathbb{Z}$. Thus we need to check whether:

$$\langle p^{e-1} a \rangle \neq \langle p^{e-1} b \rangle \text{ are non-trivial}$$

in the Lie ring.

Groups of orders p^6 and p^7 for $p \geq 7$

The first fully accurate classification of these groups was obtained by Newman, O'Brien, and Vaughan-Lee a decade ago.

Their approach is to classify nilpotent Lie rings of the same orders and then apply the Lazard correspondence. In their papers, they give presentations for the Lie rings.

By the remark on the last slide, this is enough to apply our criterion. So it is possible to determine all Beauville groups of order p^6 or p^7, provided that $p \geq 7$.

A sample of cases of order p^6

$\langle a, b \mid p^2 a, pb, p\text{-class } 3 \rangle$. Not Beauville

$\langle a, b \mid p^2 a, pb - [b, a, a], p\text{-class } 3 \rangle$. Beauville

$\langle a, b \mid pb - [b, a], p\text{-class } 3 \rangle$. Beauville

$\langle a, b \mid pa - [b, a, a, a], pb - [b, a, a, a], [b, a, a, b], [b, a, b, b] + [b, a, a, a], p\text{-class } 4 \rangle$. Not Beauville

Groups of maximal class of order at most p^p

Theorem (Blackburn, 1958)

Let G be a p-group of maximal class of order $\leq p^{p+1}$. Then $\exp G/Z(G) = p$.

As a consequence, $\exp G = p$ or p^2 and $G^p \leq Z(G)$.

Corollary

A p-group of maximal class of order $\leq p^p$ is a Beauville group if and only if $p \geq 5$ and $\exp G = p$.

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Definition

A p-group is thin if the following two conditions hold:

(i) $N \lhd G \implies \gamma_{i+1}(G) \leq N \leq \gamma_i(G)$ for some i.

(ii) $|\gamma_i(G) : \gamma_{i+1}(G)| \leq p^2$ for all i.

The basic theory of thin p-groups was established by Brandl, Caranti, Mattarei, Newman, and Scoppola in the 90's.

Examples of thin p-groups

(i) p-groups of maximal class.

(ii) All non-trivial quotients of the Nottingham group over $\mathbb{F}_p$, with p odd.

(iii) All non-trivial quotients of the binary p-adic group, a Sylow pro-p subgroup of $GL_2(\mathbb{Z}_p)$.

The criterion fails for p-groups of maximal class

There exists only one infinite pro-p group P of maximal class, and:

(i) $P = \langle s \rangle \ltimes A$, where s is of order p and $A \cong \mathbb{Z}_p^{p-1}$.

(ii) All elements in $P \smallsetminus A$ are of order p.

(iii) Every finite quotient of P is a Beauville group for $p \geq 5$.

(iv) Every finite quotient of P can be modified so that all elements outside the abelian maximal subgroup are of order p^2.
This new group is not Beauville.

(v) Let G be as in (iii) or (iv). If $|G| = p^{k(p-1)+r}$ with $k \geq 1$ and $2 \leq r \leq p$, then $\exp G = p^{k+1}$ and $|G^{p^k}| = p^{r-1}$.

Corollary

In the criterion for a p-group with 'nice power structure' to be Beauville, both implications fail to be true for infinitely many p-groups of maximal class.

Theorem

There are no Beauville p-groups of maximal class for $p = 2$ or 3.

Thus in the rest of the analysis of p-groups of maximal class, we assume that $p \geq 5$.

Also, since we have already considered p-groups of maximal class of order p^n with $n \leq p$, we will assume $n \geq p + 1$.

Definition

If G is a p-group of maximal class, we set $G_1 = C_G(G'/\gamma_4(G))$. It is a maximal subgroup of G.

The existence of Beauville structures in groups of maximal class will depend on the orders of the elements outside G_1.

Lemma (Blackburn, 1958)

Let G be a p-group of maximal class, and let M be a maximal subgroup of G. Then:

(i) *All elements in $M \smallsetminus \Phi(G)$ have the same order.*

(ii) *If $M \neq G_1$ then that order is either p or p^2.*

We call a difference $M \smallsetminus \Phi(G)$, with $M \neq G_1$, a maximal branch.

Beauville structures in p-groups of maximal class

Theorem

Let G be a p-group of maximal class of order p^n, with $n \geq p + 1$, and assume that either G is metabelian or G_1 is of class at most 2. Then one of the following holds:

(i) *There is at most one maximal branch with elements of order p, and G is not a Beauville group.*

(ii) *All maximal branches have elements of order p, and G is a Beauville group.*

(iii) *There are exactly 2 maximal branches with elements of order p, and then G is a Beauville group if and only if either n is not of the form $k(p-1)+2$, or if $n = p + 1$ and $\exp G_1 = p$.*

Definition

The Nottingham group is the group of normalised automorphisms of the ring $\mathbb{F}_p[[t]]$ of formal power series, under composition:

$$\mathcal{N}(\mathbb{F}_p) = \{f \in \operatorname{Aut} \mathbb{F}_p[[t]] \mid f(t) \equiv t \pmod{t^2}\}.$$

We will always assume that $p > 2$. Also, since p will be fixed, we write $\mathcal{N}$ for $\mathcal{N}(\mathbb{F}_p)$.

Some properties of the Nottingham group

If $\mathcal{N}_i = \{f \in \mathcal{N} \mid f(t) \equiv t \pmod{t^{i+1}}\}$, then:

- $\mathcal{N}_1 = \mathcal{N}$ and $\mathcal{N}_i \lhd \mathcal{N}$.
- $|\mathcal{N}_i : \mathcal{N}_{i+1}| = p$ and $|\mathcal{N}/\mathcal{N}_i| = p^{i-1}$.
- $\mathcal{N} \cong \lim_{i\in\mathbb{N}} \mathcal{N}/\mathcal{N}_i$. Thus $\mathcal{N}$ is a pro-p group.
- The lower central series of $\mathcal{N}$ is given by the series $\{\mathcal{N}_i\}_{i\geq 1}$, where the subgroups $\mathcal{N}_i$ with $i \equiv 2 \pmod p$ are to be deleted.
- The lower central factors of $\mathcal{N}$ are either cyclic of order p or diamonds $\mathcal{N}_i/\mathcal{N}_{i+2} \cong C_p \times C_p$ with $i \equiv 1 \pmod p$.

Theorem (Klopsch, 2000)

If $1 \neq \mathcal{W} \lhd \mathcal{N}$, either $\mathcal{W} = \mathcal{N}_i$ for some i or $\mathcal{W}$ corresponds to an intermediate subgroup of a diamond.

Thus every non-trivial quotient of $\mathcal{N}$ is a thin p-group. We will determine when it is a Beauville group.

Theorem (York, 1990; Klopsch, 2000)
$\mathcal{N}$ can be topologically generated by two automorphisms α and β of order p:

$$\alpha(t) = t(1-t)^{-1} \quad \textit{and} \quad \beta(t) = t(1-t^2)^{-1/2}.$$

We set $k(m) = p^m + \cdots + p + 1$ for every $m \geq 0$.

Theorem (York, 1990)
$\mathcal{N}^{p^m} = \mathcal{N}_{k(m)}$.

The key to determining what quotients of $\mathcal{N}$ are Beauville groups is to study first the case of $G = \mathcal{N}/\mathcal{N}_{k(m)+2}$, where $m \geq 1$.

We write a and b for the images of α and β in G, and

$$\mathfrak{M} = \{M \max G \mid a \notin M,\ b \notin M\}.$$

Lemma

(i) $\exp \Phi(G) = p^m$.

(ii) *If $M \in \mathfrak{M}$ and $g \in M \smallsetminus \Phi(G)$, then g is of order p^{m+1}.*

Let $\mathfrak{I}$ be the set of the $p+1$ (strictly) intermediate subgroups of the diamond $\mathcal{N}_{k(m)}/\mathcal{N}_{k(m)+2}$.

Theorem

The map

$$\begin{array}{ccc} \mathfrak{M} & \longrightarrow & \mathfrak{I} \\ M & \longmapsto & M^{p^m} \end{array}$$

is injective.

Theorem

The quotient $\mathcal{N}/\mathcal{N}_{k(m)+2}$ is a Beauville group for every $m \geq 1$.

Of course, if $m = 0$ then the quotient $\mathcal{N}/\mathcal{N}_3$ of order p^2 is a Beauville group if and only if $p \geq 5$.

Theorem (F-A, Gül)

(i) *If $p \geq 5$ then all quotients $\mathcal{N}/\mathcal{N}_i$ with $i \geq 3$ are Beauville groups, except for those of the form $i = k(m) + 1$.*

(ii) *If $p = 3$ then all quotients $\mathcal{N}/\mathcal{N}_i$ with $i \geq 6$ are Beauville groups, except for those of the form $i = k(m) + 1$.*

We have also determined which quotients $\mathcal{N}/\mathcal{W}$, where $\mathcal{W}$ is an intermediate subgroup of a diamond, are Beauville groups.

Corollary

The quotients of the Nottingham group over $\mathbb{F}_3$ provide Beauville groups of order 3^n for every $n \geq 5$.

This is the first explicit infinite family of Beauville 3-groups which is known.

Finally, we study Beauville structures in metabelian thin p-groups, other than groups of maximal class.

This is joint work wih Norberto Gavioli and Carlo Scoppola.

Theorem (Brandl, Caranti, Scoppola, 1992)

Let G be a metabelian thin p-group which is not of maximal class. Then $|G| \leq p^{2p}$ and G is of class at most $p+1$.

Since we have a criterion for groups of class $< p$, we can focus on metabelian thin p-groups of class p or $p+1$.

If $p = 3$ then $|G| = 3^5$ or 3^6, and a computer search shows that there are exactly 3 isomorphism classes of Beauville groups. So we assume $p \geq 5$ in the sequel.

The analysis of the general case relies on the following special case:

(i) G is of class p.

(ii) $\gamma_p(G)$ is a diamond.

Lemma

If G is as above, the following holds:

(i) $G^p \leq \gamma_p(G)$.

(ii) $|M^p| \leq p$ *for every maximal subgroup* M *of* G.

(iii) *No more than 3 maximal subgroups of G can have the same pth power.*

Theorem

Let G be as above. Then G has a Beauville structure in which one of the two triples has all elements of order p^2.

Metabelian thin p-groups: the general case

Theorem (F-A, Gavioli, Gül, Scoppola)

Let G be a metabelian thin p-group, where $p \geq 5$. Assume that G is of class $\geq p$, and not of maximal class. Then G is a Beauville group if and only one of the following cases holds:

(i) *G is of class $p+1$.*

(ii) *G is of class p and $|\gamma_p(G)| = p^2$.*

(iii) *G is of class p, $|\gamma_p(G)| = p$ and $G^p = \gamma_{p-1}(G)$.*

(iv) *G is of class p, $|\gamma_p(G)| = p$, $G^p = \gamma_p(G)$, and G has at least three maximal subgroups of exponent p.*

You can go and see Şükran Gül's poster at the permanent poster session.

In particular, she gives a second explicit infinite family of Beauville 3-groups.