THE RELATIONSHIP BETWEEN THE UPPER AND LOWER CENTRAL SERIES: A SURVEY

Martyn R. Dixon[1]

[1]Department of Mathematics
University of Alabama

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Thanks to Leonid Kurdachenko for his notes on this topic

Preliminaries

Let

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- Every non-abelian free group F satisfies $\gamma_\omega(F) = 1$ but has trivial centre.
- (Smirnov, 1953) If G is a group and $Z_\omega(G) = G$, then $\gamma_{\omega+1}(G) = 1$

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Let G be a group and let $C \leq Z(G)$, the centre of G. Suppose that G/C is finite. Then G', the derived subgroup of G, is finite.

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- This result first appeared in this form in B. H. Neumann, Proc. London Math. Soc. (3) 1 (1951), pages 178-187
- Is a corollary of an earlier more general result R. Baer, Trans. American Math. Soc. 58 (1945), 348-389

Schur's Theorem-quantitative version

Theorem (J. Wiegold, 1956, 1965)

Let G be a group. Suppose that $G/Z(G)$ is finite and $|G/Z(G)| \leq t$. Then

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- *For each prime p and each integer* $n \geq 2$ *there exists a p-group G with* $|G/Z(G)| = p^n$ *and* $|G'| = p^{1/2n(n-1)}$.

General results of Schur type

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- And if $G/Z(G)$ is periodic then G' need not be periodic.

Examples

- (A. Olshanskii) There is a group G such that $G = G'$; $Z(G)$ is free abelian of countable rank, and $G/Z(G)$ is an infinite p-group whose proper subgroups have order the prime p. ie. $G/Z(G)$ is a Tarski monster.

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- (S. I. Adian, 1971) There is a torsion-free group G such that $G/Z(G)$ is an infinite finitely generated p-group of finite exponent.

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This result builds on earlier work of A. Lubotzky, A. Mann (finite case), S. Franciosi, F. de Giovanni, L. Kurdachenko (soluble case). G is generalized radical if it has an ascending series whose factors are either locally nilpotent or locally finite. G is locally generalized radical if every finitely generated subgroup of G is generalized radical.

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Let p be a prime. The group G has finite section p-rank r if every elementary abelian p-section of G is finite of order at most p^r and there is an elementary abelian p-section of G precisely of order p^r.

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- (A. Ballester-Bolinches, S. Camp-Mora, L. Kurdachenko, J, Otal, 2013) Let G be locally generalized radical and suppose that $G/Z(G)$ has section p-rank at most s, for the prime p. Then G' has section p-rank at most $\beta_2(s)$.

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- Among the many interesting corollaries there is: Let $G/Z(G)$ be locally finite with min-p, for all primes p. Then G' is locally finite with min-p for all primes p

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- *β is defined recursively by $\beta(t, 1) = t^m = w(t)$, where $m = 1/2(\log_2 t - 1)$ and $\beta(t, k) = w(\beta(t, k-1)) + t\beta(t, k-1)$.*

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- *If $G^{\mathfrak{N}}$ is the nilpotent residual of G then $G^{\mathfrak{N}}$ is finite and $G/G^{\mathfrak{N}}$ is nilpotent.*

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- When is $G/G^{\mathbf{L}\mathfrak{N}}$ locally nilpotent?

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- (Kurdachenko, Subbotin 2013) *Let G be a group and suppose that $G/Z_\infty(G)$ is finite of order t, where $Z_\infty(G) = Z_k(G)$, for some natural number k. Then*

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- *$G/G^{\mathfrak{N}}$ is nilpotent of nilpotency class at most $\beta_4(k, t)$.*

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- there are versions of many of these results for finite groups. See N. Makarenko (2000) and L. A. Kurdachenko, A. A. Pypka and N. N. Semko (2014)

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- *(Kurdachenko, Otal and Pypka 2015) Suppose that $G/Z_k(G)$ is locally finite of exponent e. Then $\gamma_{k+1}(G)$ is locally finite of exponent at most $\beta_6(e,k)$*

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- *(C. Casolo, U. Dardano, S. Rinauro, 2016) If G has a finite normal subgroup L such that G/L is hypercentral then $|G/Z_\infty(G)| \leq |Aut(L)| \cdot |Z(L)|$*

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- *Let G be a group. Suppose that $G/Z_\infty(G)$ is locally finite and has finite rank r. Then $G^{\mathbf{L}\mathfrak{N}}$ is locally finite, $\Pi(G^{\mathbf{L}\mathfrak{N}}) \subseteq \Pi(G/Z_\infty(G))$, and there exists a function τ_4 such that $r(G^{\mathbf{L}\mathfrak{N}}) \leq \tau_4(r)$. Moreover, $G/G^{\mathbf{L}\mathfrak{N}}$ is hypercentral.*

Further Generalizations

- Let G be a group. Suppose that the hypercentre of G contains a G-invariant subgroup Z such that G/Z is a Chernikov group. Then $G^{\mathbf{L}\mathfrak{N}}$ is also Chernikov. Furthermore, $G/G^{\mathbf{L}\mathfrak{N}}$ is hypercentral.

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- Let G be a group. Suppose that the hypercentre of G contains a G-invariant subgroup Z such that G/Z is locally finite and has exponent e. Then the locally nilpotent residual, $G^{\mathbf{L}\mathfrak{N}}$, of G is locally finite of exponent at most $\beta_7(e)$, for some function β_2 depending upon e only.

Example

- There is a group $G = B \rtimes C$, where B is an infinite elementary abelian p-group and C is an infinite dihedral group such that $B = Z_\infty(G)$, G/B is polycyclic and such that G contains no normal subgroup P with P of finite rank and G/P locally nilpotent.

Some Remarks Concerning the Proofs: Theorem A

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Set $K/Z_1 = \gamma_k(G/Z_1)$, let $L = \gamma_k(G)$.Then $K = LZ_1$, $K' = L'$, $[K, G] = [L, G] = \gamma_{k+1}(G)$.

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Apply Theorem 1 to L, since $r_p(L/(L \cap Z_1) \leq \tau(r, k-1)$; we get $r_p(L') \leq \lambda_2(\tau(r, k-1))$.

Some Remarks Concerning the Proofs: Theorem A

Let $1 = Z_0 \leq Z_1 \leq \cdots \leq Z_{k-1} \leq Z_k = Z$ be the upper central series of G. Use induction on k. The case $k = 1$ is covered by Theorem 1.

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Also turns out that $r_p(\gamma_{k+1}(G)/L') \leq \theta(r, \tau(r, k-1))$, for some function θ so

$r_p(\gamma_{k+1}(G)) \leq \lambda_2(\tau(r, k-1)) + \theta_(r, \tau(r, k-1)) = \tau(r, k)$.

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G/R is locally nilpotent and $R = G^{\mathbf{L}\mathfrak{N}}$.