Groups in which each subgroup is commensurable to a normal one

dedicated to the memory of Mario Curzio

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joint work with
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B.H.Neumann's very celebrated Theorem (1955)

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Let G be a group, if
(PILF) all periodic images of G are locally finite,
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Open questions: - what about locally graded instead of PILF??
- does there exists an infinite, finitely generated, residually finite group, in which every subgroup is either finite or has finite index?

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(FA&CF) How can one put both theorems in the same framework?

Both FA and CF-groups have property:

(2-sbyf) $\forall H \leq G \quad \exists S \lhd S^G \lhd G \quad : S \leq H \ \& \ |H : S| < \infty$

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Let G be a locally finite sbyf-group. Then

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i) G is (locally nilpotent)-by-finite.

C. Casolo, Groups in which all Subgroups are Subnormal-by-Finite, Advances in Group Theory and Applications, **1** (2016)

ii) G is nilpotent-by-Chernikov (hence soluble-by-finite).
iii) G is d-sbyf for some $d \in \mathbb{N}$.

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introducing CN-groups

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Problem (IGT 2012)

- When is a CN-group *finite-by-abelian-by-finite* (and finite-by-CF)?

a trivial sufficient condition for a group to be CN.

$$\text{finite-by-}(G\text{-hamiltonian})\text{-by-finite} \Rightarrow \text{CN}$$

That is, if G has a normal series $G_0 \lhd G_1 \lhd G$, with G_0 and G/G_1 finite and each subgroup of G_1/G_0 is G-invariant, then G is a CN-group. □

Elementary examples of CN-groups

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Examples of soluble CN-groups, which are neither FA nor CF

1) $G = E \ltimes \langle\gamma\rangle$ where E is an infinite extraspecial group of exponent a prime $p \neq 2$ and $\gamma \in \mathrm{Aut}(G)$ acting as the inversion map on E/E'

2) there is 2-group G with a series $G_0 \leq G_1 \leq G$ such that G_0 and G_1 have order 2 and $x \in G \setminus G_1$ acts as the inversion map on G_1/G_0 which is the product of infinitely many cyclic groups with order 4.

MAIN RESULTS

Theorem 1

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Theorem 2

Let G be a finite-by-abelian-by-finite group.
Then:
i) G is CN if and only if it is finite-by-CF
ii) if G is CN, then the FC-center F of G has finite index in G and F' is finite.

Description of locally finite CF-groups

For abelian-by-finite CF-groups, we have results in both:
- S. Franciosi, F. de Giovanni, M.L. Newell, Groups whose subnormal subgroups are normal-by-finite, *Comm. Alg.* **23(14)** (1995), and
- J. T. Buckley, J.C. Lennox, B. H. Neumann, H. Smith, J. Wiegold, Groups with all subgroups normal-by-finite. *J. Austral. Math. Soc.* Ser. A **59** (1995).

A locally finite group G is CF iff it has a normal abelian finite index subgroup $A = C \times (D \times E)$ such that
- $\pi(C) \cap \pi(DE) = \emptyset$ (and $\pi(D) = \pi(E)$ is finite),
- D is divisibile with finite total rank,
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In particular, a locally finite CF-group has the stronger property (BCF) $\exists n \in \mathbb{N} \;:\; \forall H \leq G \;\; |H : H_G| \leq n$ (boundedly CF)

The non-periodic group $(C_\infty \times C_{p^\infty}) \rtimes \langle(+1,-1)\rangle$ is CF, but not BCF.

Among abelian-by-finite CF-groups, those which are BCF are easily detected.

IGT2012

A non-periodic abelian-by-finite CF-group G is BCF iff
there is a normal abelian subgroup B such that:
1) G acts on B by means of power automorphism (± 1),
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IGT2012

Let G be an abelian-by-finite group. If G is CN, then G is CF.

A more complete statement for CF-groups.

IGT2012

An abelian-by-finite non-periodic group G is CF iff either:
- it is elementary CF (i.e. G-hamiltonian-by-finite) or
- there is a normal abelian finite index subgroup A and a G-series

$$1 \leq V \leq A$$

1) G/V is a periodic CF group,
2) V is finitely generated free-abelian and G is power ± 1 on V.

Moreover G is BCF iff (1), (2) hold and there is $B \leq A$ such that

3) G/B has finite exponent and G acts on B as power ± 1.

boundedly CN-groups

A group is said a BCN-group if and only if
(BCN) $\exists n \in \mathbb{N} \;\; \forall H \leq G \; \exists N \lhd G \;\; : |HN : (H \cap N)| \leq n.$

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Proposition

Let G be an abelian-by-finite group.
i) if G is CN, then G is CF;
ii) if G is BCN, then G is BCF.

Recall that group is *locally graded* if every finitely generated nontrivial subgroup has a nontrivial finite image.

H.Smith, J.Wiegold, Locally graded groups with all subgroups normal-by-finite, *J. Austral. Math. Soc.* Ser. A **60** (1996).

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Theorem 3

A locally graded BCN-group is finite-by-abelian-by-finite.

Note that a finite-by-abelian-by-finite group is BCN if and only if it is finite-by-BCF

for a group G acting on an group A, consider:
(P) $\forall H \leq A \quad H = H^G$ (say that Γ is power on A, Cooper);
(AP) $\forall H \leq A \quad |H/H_G| < \infty$; (Franciosi, de Giovanni, Newell, 1995)
(BP) $\forall H \leq A \quad |H^G/H| < \infty$ (Casolo, 1988) ;
(CP) $\forall H \leq A \; \exists N = N^G \leq A : |HN : (H \cap N)| < \infty$

Tool: automorphisms of abelian groups

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U. Dardano, S. Rinauro, Inertial automorphisms of an abelian group, *Rend. Sem. Mat. Univ. Padova* **127** (2012).

1) If A is abelian and G is *finitely generated*
then properties (AP), (BP), (CP) are equivalent.
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Theorem 4

Let G be a group acting of an abelian A.
The following are equivalent:
c) G is (CP) on A.
a) G is (AP) on A/A_0 for a finite G-subgroup A_0 of A, (finite-by-AP).
b) G is (BP) on a finite index G-subgroup A_1 of A, (BP-by-finite).

Theorem 1

Let G be a group in which every periodic image is locally finite.
If G is CN, then G is finite-by-abelian-by-finite.

tools for the proof:
- by Heineken's result we reduce to p-groups
- by Casolo's result we reduce to solvable groups
- by CP-automorphisms we reduce to nilpotent groups
- then we apply Möhres's tecniques.

Theorem 2

Let G be a finite-by-abelian-by-finite group. Then:
i) G is CN if and only if it is finite-by-CF
ii) if G is CN, then the FC-center F of G has finite index in G and F' is finite;

outline of the proof:
- apply results for automorphism of abl grps

THANK YOU

for your attention