Groups in which every subgroup is commensurable to a subnormal subgroup

carlo casolo

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For a subgroup H of the group G the following are equivalent:

- H is *commensurable to a subnormal subgroup*: there exists $S \trianglelefteq\trianglelefteq G$ such that

$$[S : S \cap H] \text{ and } [H : S \cap H] \text{ are finite.}$$

- H is *subnormal-by-finite*: there exists $S \leq H$ such that

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- We say that a group G is **sbyf** if every $H \leq G$ satisfies (one of) these conditions.

- J.T. Buckley, J.C. Lennox, B.H. Neumann, H. Smith, J. Wiegold, *Groups with all subgroups normal-by-finite*. J. Austral. Math. Soc. (1995)
- G. Cutolo, J. C. Lennox, S. Rinauro, H. Smith, J. Wiegold, *On infinite core-finite groups*. Proc. Roy. Irish Acad. (1996)

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- H. Heineken, *Groups with neighbourhood conditions for certain lattices.* Note di Matematica (1996)

Theorem (Heineken, 1996)

Let G be a sbyf group. Then

(i) *If G is a Baer group then $G \in \mathfrak{N}_1$;*

(ii) *if G is* **locally finite** *then the Hirsch-Plotkin radical of G has finite index in G.*

- Baer group = all cyclic subgroups subnormal
- $\mathfrak{N}_1$ = all subgroups subnormal

Let G be a locally finite sbyf group, $B = B(G)$ its Baer radical.

- (Heineken) may assume that G is locally nilpotent, and a p-group
- $B \in \mathfrak{N}_1$, hence nilpotent by Černikov (C.),
- (Khukhro-Makarenko) $\to$ reduce to B nilpotent and G/B elementary abelian
- (Möhres Lemma) show that G/B has finite rank

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Theorem

Let G be a locally finite sbyf group. Then

- *(i) G is nilpotent-by-Černikov;*
- *(ii) there exists an integer $d \geq 1$ such that every subgroup of G admits a subgroup of finite index which is subnormal of defect at most d in G*

Question

Is a locally finite sbyf group $\mathfrak{N}_1$-by-finite?

(the answer may well be NO)

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Is a locally finite sbyf group $\mathfrak{Y}_1$-by-finite?

(the answer may well be NO)

G a p-group, $A = G'$ abelian, $G/A \simeq C_{p^\infty}$

$$\left.\begin{array}{l} A \text{ has no proper supplements in } G \\ G \text{ is Baer} \end{array}\right\} \Rightarrow G \text{ not nilpotent and } \mathfrak{Y}_1.$$

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$$\left.\begin{array}{l} A \text{ has no proper supplements in } G \\ G \textbf{ is not } \text{Baer} \end{array}\right\} \Rightarrow G \text{ sbyf not } \mathfrak{N}_1\text{-by-finite.}$$

Conjecture

A locally finite group G is nilpotent-by-Černikov if and only if every subgroup of G is subnormal-by-Černikov.

(with an appropriate definition of "subnormal-by-Černikov")

Of course, Tarski monsters are sbyf groups;

for locally graded groups the same obstructions occur as in the *CF* case – H. Smith, J. Wiegold, *Locally graded groups with all subgroups normal-by-finite*. J. Austral. Math. Soc. (1996).

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Lemma

Let G be a locally graded sbyf *group. Then there exists a normal subgroup N such that*

- *N is finitely generated and*
- *G/N is locally nilpotent-by-finite.*

Locally nilpotent-by-finite sbyf groups are $\mathfrak{N}_1$-by-nilpotent-by-Černikov

In fact:
a locally nilpotent-by-finite sbyf *group is metanilpotent-by-Černikov*

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Theorem

A torsion-free locally nilpotent-by-finite sbyf group is nilpotent-by-finite.

Let $d \geq 1$. We say that the group G is d-sbyf if for every $H \leq G$ there exists $S \leq H$ such that

$|H : S| \leq \infty$ and S is subnormal of defect at most d in G.

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By Theorem 2 every locally finite sbyf group is d-sbyf for some $d \geq 1$. Is this true for locally nilpotent-by-finite sbyf groups ?

Conjecture

There exists $f : \mathbb{N} \to \mathbb{N}$ such that if G is a locally finite d-sbyf group then G has a normal subgroup N with G/N Černikov and $\gamma_{f(d)}(N) = 1$.

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Theorem (Roseblade, 1965)

A group in which every subgroup is d-subnormal is nilpotent of class bounded by a function of d.

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Theorem (Roseblade, 1965)

A group in which every subgroup is d*-subnormal is nilpotent of class bounded by a function of* d.

Theorem (Detomi, 2004)

Let G *be a periodic group in which every subgroup has finite index in a* d*-subnormal subgroup, then* $\gamma_{\delta(d)}(G)$ *is finite.*

Let $n \geq 1$. We say that the group G is sbyf-n if for every $H \leq G$ there exists $S \leq H$ such that

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Theorem

There is a function $g : \mathbb{N} \to \mathbb{N}$ such that if G is locally graded sbyf-n group, then $|G/B(G)| \leq g(n)$.

In particular, a locally graded sbyf-n group is $\mathfrak{N}_1$-by-finite.