Free subgroups in group rings

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Let KG be the group ring of a NON-Abelian group G over a commutative ring K with 1. Let $U(KG)$ be the group of units of the group ring KG. Consider the normalized group of units

$$V(KG) = \{\sum_{g \in G} \alpha_g g \in U(KG) \mid \sum_{g \in G} \alpha_g = 1\}$$

It is well known that $U(KG) = V(KG) \times U(K)$ and $G \subseteq V(KG)$.

We can ask the following question:
Question: How "big the gap" between G and $V(KG)$?

Higman(1940) and Berman (1953) considered the question when $V(\mathbb{Z}G) = G$ in the case when G is finite?

Answer: $V(\mathbb{Z}G) = G$ if and only if G is a Hamiltonian 2-group.

Question: Let $V(\mathbb{Z}G) \neq G$. How to construct the nontrivial units in $V(\mathbb{Z}G)$ (i.e. the units from $V(\mathbb{Z}G) \setminus G$)?

Kaplansky's Conjecture: If G is a torsion free group, then

$$V(\mathbb{Z}G) = G.$$

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We shall consider only that groups G which contain elements of finite order ($\neq 1$), i.e. the group G always has a nontrivial torsion part $t(G)$.

Let $a \in t(G)$ such that $\langle a \rangle$ is not normal in G. Put

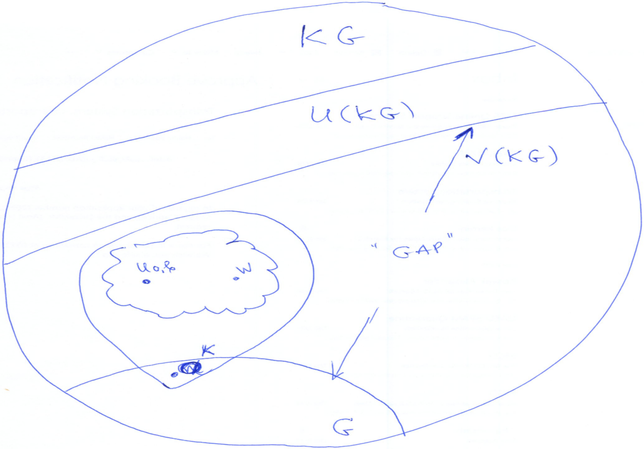
$$\widehat{a} = 1 + a + a^2 + \cdots + a^{|a|-1} \in KG.$$

Then there exists $b \in G \setminus N_G(\langle a \rangle)$ such that

$$x = (1-a)b\widehat{a} \neq 0 \qquad \text{and} \qquad x^2 = 0.$$

So $u_{a,b} = 1 \pm x$ is a nontrivial unit of infinite order in $\mathbb{Z}G$ which is called the bicyclic unit.

Consequently "the gap" between G and $V(\mathbb{Z}G)$ is "infinite".



$$U(KG) = V(KG) \times U(K)$$

$$u_{a,b} = 1 + (a-1)\, b\, \hat{a}$$

In their classic paper, B. Hartley and P. F. Pickel proved that if $G \neq V(\mathbb{Z}G)$ and G is a finite group, then $V(\mathbb{Z}G)$ contains a free subgroup.

A fundamental result was published by A. Salwa, who proved that two noncommuting unipotent elements $\{1 + x, 1 + x^*\}$ of $\mathbb{Z}G$ always generate a free group of rank 2, where x is a nilpotent element and $*$ is the classical involution of $\mathbb{Z}G$.

Consequently "the gap" between G and $V(\mathbb{Z}G)$ is really "huge".

As an example for the unipotent element $1 + x$ in Salwa's paper, one can take the bicyclic unit

$$u_{a,b} = 1 + (a-1)b\widehat{a} \in V(\mathbb{K}G).$$

This example raises the following problem.

Problem When does a unit $w \in V(\mathbb{K}G)$ exist with the property that $\langle u_{a,b}, w \rangle$ contains a free subgroup for fixed $a, b \in G$?

This problem was studied in several papers by A. Dooms, J. Goncalves, R.M. Guralnick, E. Jespers, V. Jiménez, L. Margolis, Z. Marciniak, D. Passman, A. del Rio, M. Ruiz and S. Sehgal. As a consequence, the literature of this problem is quite voluminous!

Problem. When does a unit $w \in V(\mathbb{K}G)$ exist with the property that $\langle u_{a,b}, w \rangle$ contains a free subgroup for fixed $a, b \in G$?

Answer. For fixed $a, b \in G$, it is enough to choose $w = a^k \in G$.

$$\langle u_{a,b}, a^k \rangle \supset \text{"free group"} = f(k) \in \{C_\infty \star C_\infty, \text{ or } C_n \star C_n\}.$$

Let $a \in t(G)$ such that $\langle a \rangle$ is not normal in G. Put

$$\widehat{a} = 1 + a + a^2 + \cdots + a^{|a|-1} \in KG.$$

Then there exists $b \in G \setminus N_G(\langle a \rangle)$ such that

$$x = (1-a)b\widehat{a} \neq 0 \qquad \text{and} \qquad x^2 = 0.$$

The element

$$w_k = a^k + (1-a)b\widehat{a} = \Big(1 + (1-a)b\widehat{a}\Big)a^k \qquad (1 \leq k \leq |a|)$$

is a nontrivial unit, because:

$$w_k^{-1} = a^{-k}\Big(1 - (1-a)b\widehat{a}\Big) = a^{-k} - (1-a)a^{-k}b\widehat{a}$$

If $k = |a|$ then we have the classical bicyclic unit

$$w_{|a|} = 1 + (1-a)b\widehat{a}!!!!$$

Theorem

Let $\mathbb{K}$ be an integral domain of characteristic 0 and let G be a group which has at least one non-normal finite cyclic subgroup $\langle a \rangle$ of order $|a|$. Let $b \in G \setminus \mathfrak{N}_G(\langle a \rangle)$. Additionally let M be the smallest integer with the properties that $2 \leq M \leq |a|$ and $b \in \mathfrak{N}_G(\langle a^M \rangle)$.
Let $1 \leq k < |a|$ with the property that $b \notin \mathfrak{N}_G(\langle a^k \rangle)$. The elements $\mathfrak{u}_k = a^k + (a-1)b\widehat{a}$ and $\mathfrak{z}_k = w^{-1}\mathfrak{u}_k w$ are units in $\mathbb{K}G$, where $w = 1 + (a-1)b\widehat{a} \in V(\mathbb{K}G)$ and the following hold:

(i) if $(k, |a|) = 1$, then $\langle \mathfrak{u}_k, \mathfrak{z}_k \rangle \cong C_{|a|} \star C_{|a|}$;

(ii) if $(k, |a|) \neq 1$, then $\langle \mathfrak{u}_k, \mathfrak{z}_k \rangle \cong C_s \star C_s$, where $s = \frac{|a|}{(k,|a|)}$ if $(k, M) = 1$ and $M \neq |a|$, otherwise

$$\langle \mathfrak{u}_k, \mathfrak{z}_k \rangle \cong C_\infty \star C_\infty.$$

Several problems in group theory and the theory of small dimensional topology can be reduced to the question whether a given group can be normally generate by a single element. In particular the Relation Gap problem, Wall's $D2$ Conjecture, the Kervaire Conjecture, Wiegold's Problem, Short's Conjecture and the Scott-Wiegold Conjecture (for example, see the Kourovka Book, Questions 5.52, 5.53 and 17.94).

The maim consequence of this Theorem is a solution of very old unsolved problem in the theory of integral group rings:

Problem

Let G be a finite group. When does $\mathbb{Z}G$ contains a subgroup

$$C_n \star C_n?$$

Corollary

Let $\mathbb{K}$ be an integral domain of characteristic 0 and let G be a group which has at least one non-normal finite cyclic subgroup $\langle a \rangle$ of order $|a|$. Then the group of units $U(\mathbb{K}G)$ of the group ring $\mathbb{K}G$ always contains the free product $C_{|a|} \star C_{|a|}$ as a subgroup which is normally generated by a single element.

W. Dison and T. R. Riley introduced a family of one-relator groups

$$\mathfrak{H}_r(x, y) = \langle x, y \mid (x, \underbrace{y, y, \dots, y}_{r}) = 1 \rangle, \qquad (r \geq 1)$$

that are called *Hydra* groups.

These groups are cyclic extensions of a non-abelian free group. G. Baumslag and R. Mikhailov proved that the Hydra groups (similarly to free groups) are residually torsion-free nilpotent.

Problem
When does $V(\mathbb{K}G)$ contain the Hydra group $\mathfrak{H}_r(x, y)$ as a subgroup for $r \geq 3$?

Theorem

Let $\mathbb{K}$ be an integral domain of characteristic 0 and let G be a group which has at least one non-normal finite cyclic subgroup $\langle a \rangle$ of order $|a|$. Let $b \in G \setminus \mathfrak{N}_G(\langle a \rangle)$. Additionally let M be the smallest integer with the properties that $2 \leq M \leq |a|$ and $b \in \mathfrak{N}_G(\langle a^M \rangle)$.

Let $1 \leq k < |a|$ with the property that $b \notin \mathfrak{N}_G(\langle a^k \rangle)$. Put $H_k = \langle\, 1 + (a-1)b\widehat{a},\ a^k \,\rangle \leq V(\mathbb{K}G)$. Then the following hold:

(ii) if $(k, |a|) = 1$ then H_k is a cyclic extension of $C_{|a|} \star C_{|a|}$;

(ii) if $(k, |a|) \neq 1$ and $b \notin \mathfrak{N}_G(\langle a^k \rangle)$, then H_k is a cyclic extension of $C_s \star C_s$, where $s = \frac{|a|}{(k,|a|)}$ if $M \neq |a|$ and $(k, M) \neq 1$, otherwise it is a cyclic extension of a non-abelian free group.

Moreover, in these cases H_k is a residually torsion-free nilpotent group.