On metacyclic subgroups of finite groups

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Metacyclic subgroups
Introduction

Group = finite group.

The talk is organised around two questions concerning the influence of metacyclic Sylow subgroups and 2-generator groups on the structure of a group.

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In the sequel, p will denote a prime.

Metacyclic subgroups
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If $(|G|, p^2 - 1) = 1$ and the Sylow p-subgroups of G are metacyclic, then G is p-nilpotent (Huppert I, IV, 5.10).

Metacyclic subgroups

Introduction

Question

What can be said about the structure of a group with metacyclic Sylow p-subgroups?

Metacyclic subgroups

Sylow metacyclic groups

Sylow-metacyclic groups: groups with metacyclic Sylow subgroups.

David Chillag and Jack Sonn: «Sylow-metacyclic groups and $\mathbb{Q}$-admissibility», Israel J. Math., **40**, 307–323 (1981).

Metacyclic subgroups
Sylow metacyclic groups

A group G is said to be $\mathbb{Q}$-admissible if there exists a $\mathbb{Q}$-central division algebra containing a maximal subfield K such that K is a Galois extension of $\mathbb{Q}$ and $G(K/\mathbb{Q})$ is isomorphic to G.

M. Schacher: «Subfields of division rings, I», J. Algebra, **9**, 451–477 (1968).

Metacyclic subgroups
Sylow metacyclic groups

Theorem

A soluble group is Sylow-metacyclic if and only if it is $\mathbb{Q}$-admissible.

Jack Sonn: «$\mathbb{Q}$-admissibility of solvable groups», J. Algebra, **84**, 411–419 (1983).

Metacyclic subgroups
Sylow metacyclic groups

Theorem

Let G be a Sylow-metacyclic group. Then $G = [N]A$, where N is a normal Sylow tower subgroup of G of odd order, and either

- *G is soluble and A is a Hall subgroup of G of order 2^a3^b, or*
- *N is the largest normal subgroup of G of odd order and A is one of the following groups: M_{11}, A_7, A_7^+, $\mathrm{PSL}(2, p^i)$, $\mathrm{SL}(2, p^i)$, $\mathrm{PGL}(2, p^i)$, $\mathrm{PGL}(2, p^i)^+$, $\mathrm{PGL}(2, p^i)^-$ ($i = 1, 2$ $p^i \geq 5$), $\mathrm{PGL}^*(2, p^2)$.*
  In this case, if $q \in \pi(N) \cap \pi(A)$, then the Sylow q-subgroups of G are abelian and those of A and N are cyclic.

Metacyclic subgroups
p-Sylow metacyclic groups

V. D. Mazurov: «Finite groups with metacyclic Sylow 2-subgroups.», Sibirsk. Mat. Ž., **8**, 966–982 (1967).

Metacyclic subgroups

p-Sylow metacyclic groups

Theorem

Let G be a group with a metacyclic Sylow 2-subgroup S possesing a cyclic normal subgroup N with $|S/N| > 2$. Then G is soluble.

A. R. Camina and T. M. Gagen: «Groups with metacyclic Sylow 2-subgroups», Cand. J. Math., **21**, 1234–1237 (1969).

Metacyclic subgroups

p-Sylow metacyclic groups

Theorem

If G is a p-soluble group with metacyclic Sylow p-subgroups, then

1. *if $p = 2$, then $G/\operatorname{O}_{2',2}(G)$ is of odd order, or is isomorphic to Σ_3. In particular, the 2-length of G is at most 2.*
2. *if $p > 2$, then the p-length of G is at most 1.*

V. S. Monakhov and E. E. Gribovskaya: «Maximal and Sylow Subgroups of Solvable Finite Groups», Math. Notes, **70**, 545–552 (2001).

Metacyclic subgroups

p-Sylow metacyclic groups

Theorem (with Su and Wang)

Let G be a group with metacyclic Sylow p-subgroups. Suppose that $G/\operatorname{O}_{p'p}(G)$ is of odd order, and let $G^* = G/\operatorname{O}_{p'}(G)$. Then G^* has a normal Sylow p-subgroup P^*, and $G^* = [P^*](H_1 \times H_2)$, where H_1 is an abelian group of exponent dividing $p-1$, H_2 is a cyclic group with exponent dividing $p+1$. In particular, G' is p-nilpotent.

Metacyclic subgroups
p-Sylow metacyclic groups

Theorem (with Su and Wang)

Let G be a group with metacyclic Sylow p-subgroups. If $|G/O_{p'p}(G)|$ and $p+1$ are coprime, then G is p-supersoluble.

Metacyclic subgroups

p-Sylow metacyclic groups

Corollary (with Su and Wang)

Suppose G is a group of odd order with metacyclic Sylow p-subgroups. Then G is p-supersoluble if and only if $|G/\,\mathrm{O}_{p'p}(G)|$ and $p+1$ are coprime.

Metacyclic subgroups

p-Sylow metacyclic groups

Corollary (Berkovich)

Let the group $G = AB$ be the product of the subgroups A and B. If G is of odd order and the Sylow p-subgroups of A and B are cyclic, then G is p-supersoluble.

Metacyclic subgroups

p-Sylow metacyclic groups

Theorem (with Su and Wang)

Assume p is odd and let G be a group with metacyclic Sylow p-subgroups. Set $G^* = G/\operatorname{O}_{p'}(G)$*. Then* $(|G/\operatorname{O}_{p'p}(G)|, p-1) = 1$ *if and only if* G^* *satisfies the following properties:*

(1) *A Sylow p-subgroup of* G^* *is normal in* G^**.*

(2) *The Hall* p'*-subgroups of* G^* *are cyclic groups of odd order dividing* $p+1$*.*

Metacyclic subgroups
p-Sylow metacyclic groups

Theorem (with Su and Wang)

Suppose that a Sylow p-subgroup of a group G is metacyclic. Then G is p-nilpotent if and only if $(|G/\operatorname{O}_{p'p}(G)|, p^2-1) = 1$.

2-generator subgroups

Aim: characterise the non-nilpotent groups in which every 2-generator subgroup is metacyclic (Shirong Li's question).

Basic idea: Assume that a group G has 2-generator subgroups in a class $\mathcal{X}$ of groups which is subgroup-closed and the minimal non-$\mathcal{X}$-groups are 2-generator. Then G belongs to $\mathcal{X}$.

2-generator subgroups

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2-generator subgroups

Soluble:

P. Flavell: «Finite groups in which every two elements generate a soluble subgroup», Invent. Math., **121**, 279–285 (1995).

Supersoluble:

K. Doerk: «Minimal nicht überauflösbare, endliche Gruppen», Math. Z., **91**, 198–205 (1966).

Nilpotent:

B. Huppert: «Endliche Gruppen I», Satz III, 5.2 (1967).

2-generator subgroups

Minimal non-metacyclic p-groups:

N. Blackburn: «Generalizations of certain elementary theorems on p-groups», Proc. London Math. Soc., **11**, 1–22 (1961).

Consequence: the class $\mathcal{M}$ of all groups with 2-generator subgroups metacyclic contains non-metacyclic groups.

2-generator subgroups

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2-generator subgroups

For odd primes p, the p-groups in $\mathcal{M}$ are exactly the modular ones.

2-generator subgroups

Theorem

The class of 2-groups in $\mathcal{M}$ coincides with the class of monotone 2-groups.

A. Mann: «The number of generators of finite p-groups», J. Group Theory., **8**, 317–337 (2005); **14**, 329–331 (2011).

A group G is monotone if for every pair (H, K) of subgroups of G, $H \leq K$ implies that $d(H) \leq d(K)$.

2-generator subgroups

Classification:

E. Crestani and F. Menegazzo: «On monotone 2-groups», J. Group Theory, **15**, 359–383 (2012).

2-generator subgroups

Theorem (with Cossey)

A non-nilpotent group G has 2-generator subgroups metacyclic if and only if

1. *G is supersoluble and metabelian, with Sylow subgroups modular for odd primes and monotone groups for the prime 2.*
2. *$N = G^{\mathfrak{N}}$ (the nilpotent residual of G) is abelian (and $\neq 1$) and so $G = NH$, $N \cap H = 1$ for every Carter subgroup H of G.*

2-generator subgroups

Theorem (with Cossey)

1. *H acts on N as power automorphisms and if π is the set of primes dividing N then every Sylow p-subgroup of H is cyclic if $p \in \pi$ and if $H_{\pi'}$ is a Hall π'-subgroup of H, then $H_{\pi'}/C_{H_{\pi'}}(N)$ is cyclic.*
2. *If $q \in \pi'$, and H_q is a Sylow q-subgroup of H and $x \notin C_{H_q}(N)$ and $y \in C_{H_q}(N)$, then $K = \langle x, y\rangle = U\langle x\rangle$ with U cyclic, normal in K and contained in $C_H(N)$.*

2-generator subgroups

Let G be a group. For each isomorphism type of chief factor A (as G-module) let $\delta_G(A)$ denote the number of complemented chief factors of G isomorphic to A in a fixed chief series and let $\Omega(G)$ denote the set of non-isomorphic complemented chief factors.

2-generator subgroups

Lemma

Assume that G is supersoluble. With the notation above

$$d(G) = max_{A \in \Omega(G)} h_G(A)$$

where $h_G(A) = (\delta_G(A) + 1 - \theta_G(A))$ *and* $\theta_G(A) = 1$ *if A is a trivial G-module, and* $\theta_G(A) = 0$ *otherwise.*

A. Lucchini and M.C.Tamburini: «Minimal generation of finite soluble groups by projectors and normalizers», Glasg. Math. J., **41**, 303–312 (1999).