On Normalized Integral Table Algebras Generated by a Faithful Non-real Element of Degree 3 with an Element of Degree 2

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Table algebras

Definition

Let $B = \{b_1 = 1, ..., b_k\}$ be a distinguished basis of an associative commutative complex algebra A. A pair (A, B) is called a table algebra if it satisfies the following conditions

1. $b_i b_j = \sum_{m=1}^{k} \lambda_{ijm} b_m$ with λ_{ijm} being non-negative reals;
2. there exists a table algebra automorphism $x \mapsto \bar{x}$ of A whose order divides two such that $\overline{B} = B$ ($\bar{}$ defines a permutation on $[1, k]$ via $\overline{b_i} = b_{\bar{i}}$);
3. there exists a coefficient function $g : B \times B \to \mathbb{R}^+$ such that $\lambda_{ijm} = g(b_i, b_m)\lambda_{\bar{j}mi}$

An element b_i is called real if $i = \bar{i}$. For any $x = \sum_{i=1}^{k} x_i b_i$ we set $Irr(x) := \{b_i \in B \mid x_i \neq 0\}$.

Table subsets

Definition

A non-empty subset $T \leq B$ is called a table subset if $Irr(T\overline{T}) \subseteq T$. In this case a linear span $S := \langle T \rangle$ of T is a subalgebra of A. The pair (S, T) is called table subalgebra of (A, b).

Faithful elements

Since an intersection of table subsets is a table subset by itself, one can define a table subset generated by an element $b \in B$, notation B_b, as the intersection of all table subsets of B containing b. An element $b \in B$ with $B_b = B$ is called faithful.

Isomorphism between table algebras

Rescaling

Given a table algebra (A, B) one can replace its table basis $B = \{b_1, ..., b_k\}$ by $B' = \{\beta_1 b_1, ..., \beta_k b_k\}$ where β_i's are positive real numbers with $\beta_1 = 1$. A table algebra (A, B') is called a rescaling of (A, B).

Isomorphisms between TA

Two table algebras (A, B) and (A', B') are called isomorphic, notation $(A, B) \cong (A', B')$, if there exists an algebra isomorphism $f : A \to A'$ such that $f(B)$ is a rescaling of B'. In the case of $f(B) = B'$, the algebras are called exactly isomorphic, notation $(A, B) \cong_x (A', B')$.

Degree homomorphism

Theorem (Arad, Blau)

Let (A, B) be a table algebra. Then there exists a unique algebra homomorphism $a \mapsto |a|, a \in A$ onto $\mathbb{C}$ such that $|b| = |\bar{b}| > 0$ holds for all $b \in B$. The number $|b|$ is called the degree of b.

Normalized and standard TAs

An element $b_i \in B$ is called standard (normalized) if $\lambda_{i\bar{i}1} = |b_i|$ ($\lambda_{i\bar{i}1} = 1$). A table algebra is called standard (normalized) if all the elements of its table basis are standard (normalized). Notice that any table algebra may be rescaled to a standard or normalized one. If (A, B) is normalized, then $g(b_i, b_j) = 1$. For standard table algebras $g(b_i, b_j) = |b_i|/|b_j|$.

The order of a TA

Definition

The number

$$o(B) := \sum_{i=1}^{k} \frac{|b_i|^2}{\lambda_{i\bar{i}1}}$$

does not depend on a rescaling of (A, B) and is called the order of (A, B). If (A, B) is standard, then $o(B) = \sum_{i=1}^{k} |b_i|$. If (A, B) is normalized, then $o(B) = \sum_{i=1}^{k} |b_i|^2$.

Definition

A table algebra is called integral if all its degrees and structure constants are non-negative integers.

Examples: character algebra of a finite group

Let G be a finite group and $Ch(G)$ denote the algebra of all complex valued class functions on G with pointwise multiplication. This algebra has a natural basis $Irr(G)$ consisting of irreducible characters of G. The pair $(Ch(G), Irr(G))$ satisfies the axioms of a table algebra. In this case $\bar{\chi}, \chi \in Irr(G)$ is a complex conjugate character and the degree function of χ is a usual degree of an irreducible character - $\chi(1)$. The algebra $(Ch(G), Irr(G))$ is a normalized integral table algebra (NITA, for short).

Examples: the center of a finite group algebra

Let G be a finite group and $Z(\mathbb{C}[G])$ denote the center of a group algebra. $Z(\mathbb{C}[G])$ is a subalgebra of $\mathbb{C}[G]$. Let $C_1 = \{1\}, C_2, ..., C_k$ be a complete set of conjugacy classes of G. Denote $b_i := \sum_{g \in C_i} g$, $Cla(G) := \{b_1, ..., b_k\}$. Then $Z((\mathbb{C}[G]), Cla(G))$ satisfies the axioms of a table algebra with $\overline{b_i} = \sum_{g \in C_i} g^{-1}$ and degree function $|b_i| = |C_i|$. The algebra $Z((\mathbb{C}[G]), Cla(G))$ is a standard integral table algebra (SITA, for short).

Table algebras classification results

Minimal degree

A minimal degree $m(B)$ of an ITA (A, B) is $\min\{|b_i| \mid i > 1\}$. ITAs containing a faithful element of degree 2 with $m(B) = 2$ were classified by Blau.

Homogeneous ITAs

HITAs of degrees $1, 2, 3$ were completely classified in a series of papers by Arad, Blau, Fisman, Miloslavsky and Muzychuk.

Standard ITAs

SITAs containing a faithful non-real element of minimal degree 3 and 4 were classified in a series of papers by Arad, Arisha, Blau, Fisman and Muzychuk.

Normalized integral table algebras

Let (A, B) be a NITA. Define $L_i(B) \subseteq B$ to be the set of the elements of B of degree i.

Let (A, B) be a NITA containing a faithful element b of minimal degree m. If $m = 1$, then (A, B) is exactly isomorphic to the character algebra of a cyclic group.If $m = 2$, then the classification of such algebras follows from Blau's result. In this talk we present the results obtained for $m = 3$ under additional assumption that b_3 is non-real. By definition $L_2(B) = \emptyset$.

Normalized integral table algebras

The goal of our research was to classify NITAs (A, B) generated by a faithful non-real element of degree 3 under the assumptions that $|L_1(B)| = 1$ and $L_2(B) = \emptyset$.

Theorem (Arad, Chen)

Let (A, B) be a NITA of minimal degree 3 containing a faithful element b_3 of minimal degree 3. Then $b_3\overline{b_3} = 1 + b_8$ where $b_8 \in B$ is real of degree 8 and one of the following holds.

1. $(A, B) \cong_x ((Ch(G), Irr(G)), G \cong PSL(2, 7)$;
2. $b_3^2 = b_4 + b_5$ where $b_4, b_5 \in B$;
3. $b_3^2 = c_3 + b_6$ where $c_3, b_6 \in B$, $c_3 \neq b_3, \bar{b}_3$;
4. $b_3^2 = \overline{b}_3 + b_6, b_6 \in B$ is non-real;

Theorem (Arad, Xu)

The second case cannot occur.

The third case

Theorem (Arad, Cohen, Arisha)

Assume that

$$b_3^2 = c_3 + b_6, c_3 \neq b_3, \bar{b}_3.$$

Then $(b_3b_8, b_3b_8) = 3, 4$. If $(b_3b_8, b_3b_8) = 3$ and c_3 is real, then there exists a unique NITA of dimension 22. If c_3 is not real, then there exists a unique NITA of dimension 32 satisfying these conditions. Both NITAs are not induced from character tables of finite groups.

Problem

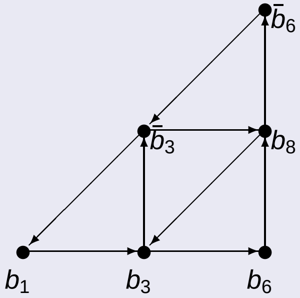
Classify the NITAs in the title with $(b_3b_8, b_3b_8) = 4$.

The fourth case

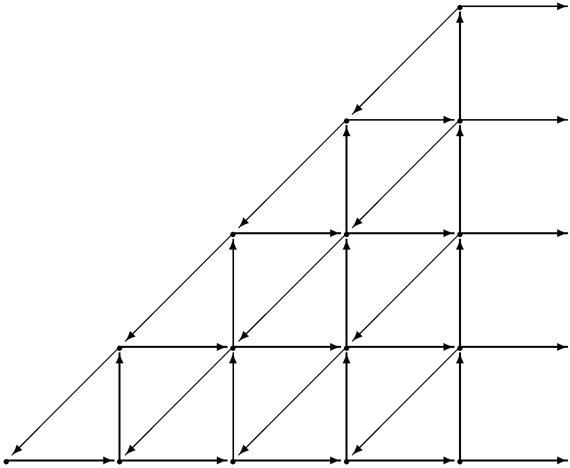
A representation graph

A representation graph of $b_i \in B$ is a weighted graph on B in which two vertices b_j and b_k are connected by an edge of weight λ_{ijk}.

A representation graph of b_3 at distance two



Graph C_n



Fourth case: $b_3^2 = \bar{b}_3 + b_6$

Definition

A NITA (A, B) in the title satisfies C_n -condition if the representation graph at distance n is isomorphic to C_n. We say that n is a stopping number for (A, B) if n is a maximal number for which (A, B) satisfies C_n-condition. In the case when (A, B) satisfies C_n-condition for each n, we say that its stopping number is ∞. In the latter case (A, B) is infinite dimensional algebra with $|B| = \aleph_0$,

Fourth case: $b_3^2 = \bar{b}_3 + b_6$

Theorem (Arad, Cohen)

a) There exist only one algebra of fourth type with stopping number two, namely $(Ch(PSL(2,7)), Irr(PSL(2,7))$.

b) If the stopping number is 3, then there exist $b_6, b_{10}, b_{15} \in B$, where b_6 is non-real, such that

$$b_3^2 = \bar{b}_3 + b_6, \bar{b}_3 b_6 = b_3 + b_{15},$$
$$b_3 b_6 = b_8 + b_{10} \text{ and } (b_3 b_8, b_3 b_8) = 3.$$

Moreover, if b_{10} is real then $(A, B) \cong_x (Ch(3 \cdot A_6), Irr(3 \cdot A_6))$ of dimension 17. In this case $(A, B) \cong_x (A, B, C_3)$ is C_3-Table Algebra. The case of b_{10} being non-real is still open.

Fourth case: $b_3^2 = \bar{b}_3 + b_6$

Theorem (Arad, Cohen)

There exists no NITA of fourth type with stopping number at least 43.

Theorem (Arad, Cohen, Muzychuk)

There exists a unique infinite dimensional NITA of fourth type with stopping number ∞. This is the NITA of polynomial characters of $SL_3(\mathbb{C})$.

Open Problem

Classify all NITAs of fourth type with stopping number in the range $[4, 42]$.

The goal of our recent research is to eliminate the assumption $L_2(B) = \emptyset$, i.e. classify NITA (A, B) generated by a faithful non-real element of degree 3 under the assumption $L_1(B) = 1$. It is well-known that a perfect group G has no non-trivial linear character. By results of H.F. Blichfeldt, H. Blau, Z. Arad, M. Awais and C. Guiyun proved the following

Lemma

Let G be a perfect finite group with a faithful irreducible character of degree 3. Then G has no irreducible character of degree 2.

This inspired Arad, Awais and Chen to state the following

Conjecture

Let (A, B) be a NITA generated by a faithful non-real element $b_3 \in B$ of degree 3. Assume that $L_1(B) = 1$. Then $L_2(B)$ is an emptyset.

In our research we proved the conjecture under additional conditions. If one can prove the conjecture then we can eliminate the assumption $L_2(B) = \emptyset$ in the Main Theorem of [1].

The first step is to prove the following

Proposition

Let (A, B) be a NITA generated by an element $b_2 \in B$ of degree 2 and $L_1(B) = 1$. Then

$$(A, B) \cong_x (Ch(SL(2, 5)), Irr(SL(2, 5)).$$

This proposition follows from Blau's classification of ITAs generated by an element of degree 2.

Remark

$SL(2, 5)$ has two irreducible faithful real characters $c_2, c_2^* \in Irr(SL(2, 5))$.

Define type (H) to be a NITA (A, B) satisfying the following conditions

1. $B = B_{b_3}, b_3 \in B$ is a non-real element of degree 3;
2. $L_1(B) = 1$;
3. $L_2(B) = \{c_2, c_2^*\}$.

By our Proposition $B_{c_2} \cong_x B_{c_2^*} \cong (Ch(SL(2,5)), Irr(SL(2,5)))$.
The goal at this point is to prove the following weak conjecture

Weak conjecture

NITA of type (H) does not exist.

If one can prove this Theorem then we have first step in order to prove our conjecture that $L_2(B) = \emptyset$.

Lemma 1

Let (A, B) be a NITA of type (H). If $a \in Irr(SL(2,5))$ then $ab_3 \in B \setminus Irr(SL(2,5))$ and $Irr(b_3^2) \subseteq B \setminus Irr(SL(2,5))$

Lemma 2

Let (A, B) be a NITA of type (H). Then $b_3\bar{b}_3 = 1 + b_8$ where $b_8 \in B$ is of degree 8.

The next step is to prove

Lemma 3

Let (A, B) be a NITA of type (H). Then $b_3\bar{b}_3 = 1 + b_8$ where $b_8 \in B$ is an element of degree 8 and one the following holds

<u>Case 1:</u> $b_3^2 = \bar{b}_3 + b_6, b_6 \in B$ is real of degree 6;

<u>Case 2:</u> $b_3^2 = \bar{b}_3 + b_6, b_6 \in B$ is non-real of degree 6;

<u>Case 3:</u> $b_3^2 = c_3 + b_6, b_6 \in B\ c_3 \neq b_3, \bar{b}_3$;

<u>Case 4:</u> $b_3^2 = c_4 + c_5, c_4, c_5 \in B$.

Lemma 4

Case 1 of Lemma 3 is impossible.

Lemma 5

In case 2 we proved that there exist $b_6, b_{10}, b_{15} \in B$, where b_6 is non-real, such that

$$b_3^2 = \bar{b}_3 + b_6, \bar{b}_3 b_6 + b_3 + b_{15}, b_3 b_6 = b_8 + b_{10},$$

and $(b_3 b_8, b_3 b_8) = 3$.

If b_{10} is real then Case 2 is impossible. If b_{10} is non-real then Case 2 is still open.

We plan to continue our efforts to prove the weak conjecture.