# Coverings of commutators in profinite groups

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Let us look at finite coverings by subgroups of $G$, so $|I| < \infty$ and we have $G = \bigcup_{i=1}^{n} H_i$, with $H_i < G$.

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What information about the structure of $G$ can be deduced from the properties of the covering subgroups $H_i$?

# Combinatorial tool

## Lemma (B.H. Neumann)

*Let* $G = \bigcup_{i=1}^{n} H_i g_i$*, where* $H_1, \dots, H_n$ *are subgroups of* $G$*. Then we can omit from the covering any* $H_i g_i$ *for which* $[G : H_i]$ *is infinite.*

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We can get rid of the cosets of subgroups of infinite index without losing the covering property.

In particular if $G = \bigcup_{i=1}^{n} H_i$ is a finite covering by subgroups, then we can assume that $[G : H_i] < \infty$.

# Some characterizations for abstract groups

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Assume that $G = \bigcup_{i=1}^{n} C_i$, where $C_i \leq G$ cyclic. If $G$ is infinite, let us prove that $G$ is cyclic.
By Neumann's lemma we can suppose for each $i$ that $[G : C_i]$ is finite.
Thus any $C_i$ is cyclic infinite and $G$ is torsion-free.

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Thus any $C_i$ is cyclic infinite and $G$ is torsion-free.
Baer's result implies that $G$ is central-by-finite, so $G'$ is finite and therefore trivial. We get the conclusion noting that $G$ is f. g. abelian torsion-free with a cyclic subgroup of finite index. The converse is obvious.

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- $\delta_0 = x_1,\ \delta_k = [\delta_{k-1}(x_1, \ldots, x_{2^{k-1}}), \delta_{k-1}(x_{2^{k-1}+1}, \ldots, x_{2^k})]$, for $k \geq 1$, then $w(G) = G^{(k)}$ (derived words)

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If the set of all $w$-values in $G$ can be covered by finitely many subgroups, i.e. $G_w \subseteq \bigcup_{i=1}^{n} H_i$, one could hope that the structure of $w(G)$ is somehow similar to that of the covering subgroups $H_i$.

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*If a locally compact Hausdorff space is a union of countably many closed subsets, then at least one of subsets has non-empty interior.*

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**Baire's Category Theorem**

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If in a profinite group $G$ the set of all $w$-values can be covered by countably many closed subgroups with specific properties, then one could hope that the structure of $w(G)$ is similar to that of the covering subgroups.

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The word $w = x_1$ is a multilinear commutator word of weight $1$. If $u, v$ are multilinear commutator words of weights $m$ and $n$ respectively involving different indeterminates, then $[u, v]$ is a multilinear commutator word of weight $m + n$.

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Examples: the lower central words $\gamma_k$ for $k \geq 1$ and the derived words $\delta_k$ for $k \geq 0$.

Not all commutator words are multilinear: Engel-words $[x_1, x_2, \overset{k}{\ldots}, x_2]$ for $k \geq 2$.

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  - subgroups of finite rank.

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- 1st type: If $G$ is a profinite group in which the set of all $w$-values can be covered by countably many subgroups with property $\mathcal{P}$, then $w(G)$ is finite-by-($\mathcal{P}$).

- 2nd type: If $G$ is a profinite group in which the set of all $w$-values can be covered by countably many subgroups with property $\mathcal{P}$, then $w(G)$ has the property $\mathcal{P}$ as well.

Here the property $\mathcal{P}$ is one of the mentioned above:
cyclic, procyclic, abelian, nilpotent, periodic, of finite rank, etc

# Examples of results of 1st type

If $G$ is a profinite group in which the set of all $w$-values can be covered by countably many subgroups with property $\mathcal{P}$, then $w(G)$ is finite-by-($\mathcal{P}$).

# Profinite groups and coverings by procyclic subgroups

**Theorem**

*A profinite group $G$ can be covered by countably many procyclic subgroups if and only if $G$ is finite-by-procyclic.*

## Commutators covered by procyclic subgroups

**Theorem**

*Let $G$ be a profinite group. The set of all commutators of $G$ is contained in a union of countably many procyclic subgroups if and only if the commutator subgroup $G'$ is finite-by-procyclic.*

- If the set of commutators of a profinite group $G$ is contained in a union of countably many procyclic subgroups, then the whole $G'$ is contained in a union of countably many procyclic subgroups.

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For pro-$p$ groups we can distinguish between groups in which commutators are covered by countably many procyclic subgroups and those in which the covering of commutators requires only finitely many procyclic subgroups.

# Coverings by abelian and nilpotent subgroups

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- In view of Hall's theorem we have: an abstract group $G$ admits a finite covering by nilpotent subgroups if and only if $G$ is finite-by-nilpotent.

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- A profinite group $G$ is covered by countably many nilpotent subgroups if and only if $G$ is finite-by-nilpotent.

A profinite group $G$ admits a countable covering by abelian (respectively nilpotent) subgroups if and only if $G$ admits a finite covering by subgroups with the respective property.

## Commutators covered by nilpotent groups

**Theorem**

*Let $G$ be a profinite group. The following conditions are equivalent.*

(1) *The set of all commutators in $G$ is covered by countably many nilpotent subgroups;*

(2) *The commutator subgroup $G'$ is finite-by-nilpotent;*

(3) *There exists a positive integer $m$ such that $Z_m(G')$ is open in $G'$.*

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Let $w$ be a multilinear commutator word and $G$ a profinite group in which the $w$-values are covered by countably many nilpotent subgroups.
Is $w(G)$ finite-by-nilpotent?

# Examples of results of 2nd type

If $G$ is a profinite group in which the set of all $w$-values can be covered by countably many subgroups with property $\mathcal{P}$, then $w(G)$ has the property $\mathcal{P}$ as well.

# Profinite groups and coverings by periodic subgroups

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A analogue result holds for a covering given by subgroups of finite rank.

# Can we bound the exponent of $w(G)$?

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However we do not know whether the exponent of $w(G)$ depends on the exponents of the covering subgroups.

Let $w$ be a multilinear commutator word and $G$ a profinite group in which all $w$-values are contained in a union of countably many subgroups of finite exponent dividing $e$ (of finite rank at most $r$, respectively). Does $w(G)$ necessarily have an open subgroup of finite exponent dividing $e$ (of finite rank at most $r$, respec.)?

# Quantitative results for the case of $\gamma_k(G)$

**Theorem**

*Let* $e, k, s$ *be positive integers and* $G$ *a profinite group that has subgroups* $G_1, G_2, \ldots, G_s$ *whose union contains all* $\gamma_k$*-values in* $G$. *Suppose that each of the subgroups* $G_1, G_2, \ldots, G_s$ *has finite exponent dividing* $e$. *Then* $\gamma_k(G)$ *has finite* $(e, k, s)$*-bounded exponent.*

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An analogue result holds for a covering given by subgroups of finite rank.

Let $e, k, s$ be positive integers and $w$ a multilinear commutator word of weight $k$. Suppose that $G$ is a profinite group having subgroups $G_1, G_2, \dots, G_s$, each of finite exponent dividing $e$ (of finite rank at most $r$, respectively), whose union contains all $w$-values in $G$.
Does $w(G)$ necessarily have finite $(e, k, s)$-bounded exponent ($(r, k, s)$-bounded rank, resp.)?

Thank you!