ISCHIA GROUP THEORY 2016, March 29th – April 2nd

Groups of rational interval exchange transformations

Agnieszka Bier and Vitaliy Sushchanskyy

Institute of Mathematics, Silesian University of Technology, Poland

agnieszka.bier@polsl.pl

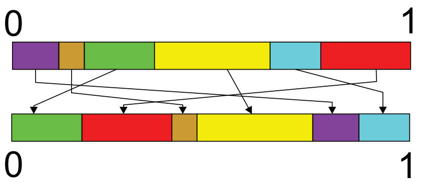
Interval exchange transformations

Let I denote the half-open interval $[0,1)$ and let

$$\pi = \{a_0, a_1, ..., a_n\}, \qquad 0 = a_0 \le a_i < a_{i+1} \le a_n = 1$$

be a partition of I into n subintervals $[a_i, a_{i+1})$, $i = 0, ..., n-1$. A left-continuous bijection $f : [0,1) \to [0,1)$ acting as piecewise translation, i.e. shuffling the subintervals $[a_i, a_{i+1})$, is called an ***interval exchange transformation (iet)*** of I. The group of all interval exchange transformations of I is denoted by IET.

The action of an exemplary iet for $n = 6$ is demonstrated on the scheme below:



Every transformation $f \in$ IET may be represented in a (unique) *canonical form* as a pair

$$f = (\pi, \sigma),$$

where π is a partition of I into n subintervals (n being the smallest possible), and $\sigma \in S_n$ is a permutation of the set of n elements.

Topology on IET

Let IET_σ denote the set of all iets with canonical form defined with a given permutation $\sigma \in S_n$. The *natural topology* on IET arises from the mutual correspondence between the subsets IET_σ, $\sigma \in S_{n+1}$, and the standard n-dimensional open simplex

$$\Delta_n = \{(x_1, x_2, ..., x_{n+1}) \in {\mathbb{R}_+}^{n+1} \mid \sum_{i=1}^{n+1} x_i = 1\},$$

which is given by:

$$\psi : \mathrm{IET}_\sigma \longrightarrow \Delta_n$$
$$\psi(\{a_0, a_1, ..., a_{n+1}\}, \sigma) = (a_1 - a_0, a_2 - a_1, ..., a_{n+1} - a_n).$$

The assumptions imply that IET is a disjoint union of all IET_σ, and hence if all of IET_σ are defined to be open, we obtain the topology on IET.

We note that IET is not a topological group with this topology, as the operation of composition of iets is not continuous.

Rational interval exchange transformations

An interval exchange transformation $f = (\pi, \sigma) \in$ IET defined by the rational partition $\pi = \{a_0, a_1, ..., a_n\}$ where all a_i are rational numbers is called *rational iet*. The subset of all rational iets is a subgroup of IET, which we denote by RIET.

Every rational interval exchange transformation may be considered as a transformation defined by the partition π_n of interval I into n subintervals of equal length.

If $f = (\pi, \sigma)$, where $\pi = \{0, a_1, ..., a_{n-1}, 1\}$, where $a_i = \frac{p_i}{q_i} \in \mathbb{Q}$, is a rational interval exchange transformation, then there exists a rational partition π_q into $q = LCM(q_1, ..., q_{n-1})$ subintervals of equal length such that f shuffles the q subintervals defined by π_q according to a permutation $\sigma' \in S_q$ such that the action of $f' = (\pi_q, \sigma')$ on I is equivalent to the action of f.

Moreover, if f and g are two rational iets, then there exists a partition π_m of the interval I into m equally sized subintervals, such that f and g act on I by shuffling the subintervals defined by π_m (it is enough to take m as the least common multiplier of all endpoints of partitions defining f and g).

Remark. RIET is a dense subgroup of IET.

Supernatural numbers

A sequence of natural numbers $\bar{n} = (n_1, n_2, ...)$ is called the ***divisible sequence***, if $n_i | n_{i+1}$ (n_i divides n_{i+1}) for every $i \in \mathbb{N}$. In the divisible sequence $\bar{n}$ the set of prime divisors of n_i is contained in the set of prime divisors of n_{i+1} and this includes also the multiplicities of these divisors. Thus with every divisible sequence $\bar{n}$ one may associate the ***supernatural number*** $\hat{n}$, defined as the formal product

$$\hat{n} = \prod_{p_i \in P} p_i^{\varepsilon_i},$$

where P denotes the (naturally ordered) set of all primes, and $\varepsilon_i \in \mathbb{N} \cup \{0, \infty\}$ for every $i \in \mathbb{N}$.

The supernatural number $\hat{n}$ associated to the divisible sequence $\bar{n}$ is called the *characteristic* of this sequence.

Subgroups of RIET defined by supernatural numbers

By $\mathrm{RIET}(n) = \{f \in \mathrm{RIET} \mid f = (\pi_n, \sigma), \sigma \in S_n\}$, $n \in \mathbb{N}$ we denote the subgroup of RIET, isomorphic to S_n. For a divisible sequence $\bar{n} = (n_1, n_2, ...)$ we define the ***diagonal embeddings***

$$\varphi_i : \mathrm{RIET}(n_i) \hookrightarrow \mathrm{RIET}(n_{i+1}),$$

where $n_i | n_{i+1}$, which correspond to the ***diagonal embeddings*** of S_{n_i} into $S_{n_{i+1}}$.

Subgroups of RIET defined by supernatural numbers

In particular, if $f = (\pi_{n_i}, \sigma) \in \mathrm{RIET}(n_i)$ and $n_{i+1} = k \cdot n_i$ then the diagonal embedding φ_i is defined by the following rule:

$$f^{\varphi_i}\left(\left[\frac{l \cdot n_i + j}{n_{i+1}}, \frac{l \cdot n_i + j + 1}{n_{i+1}}\right)\right) = \left[\frac{l \cdot n_i + \sigma(j)}{n_{i+1}}, \frac{l \cdot n_i + \sigma(j) + 1}{n_{i+1}}\right),$$

for all $j = 0, 1, ..., n_i - 1$ and $l = 0, 1, ..., k-1$.

The groups $\mathrm{RIET}(n)$ together with the diagonal embeddings form a direct system of groups. The corresponding direct limit:

$$\mathrm{RIET}(\bar{n}) = \lim_{\substack{\longrightarrow \\ i}} \mathrm{RIET}(n_i)$$

is a subgroup of RIET.

For instance, if $\hat{M} = \prod_{p_i \in P} p_i^{\infty}$ is the characteristic of the divisible sequence $\bar{m}$, then $\mathrm{RIET}(\bar{m}) = \mathrm{RIET}$.

Results

Let $\hat{n}$ be a supernatural number with characteristic $\bar{n}$. Then

- If $\hat{n}$ is infinite, then the subgroup $\mathrm{RIET}(\bar{n})$ is isomorphic to the homogeneous symmetric group $S_{\bar{n}}$.
- If $\hat{n}$ is infinite, then the subgroup $\mathrm{RIET}(\bar{n})$ is dense in RIET.
- $\mathrm{RIET}(\hat{n})$ is either finite or locally finite. In particular $\mathrm{RIET}(\hat{n})$ is finitely generated if and only if it is finite.
- If $\bar{n} = (n_1, n_2, ...)$ and $\bar{m} = (n_i, n_{i+1}, ...)$, $i > 1$, then $\mathrm{RIET}(\bar{n}) = \mathrm{RIET}(\bar{m})$.
- For every prime p the subgroup $\mathrm{RIET}(p^\infty)$ is the minimal dense subgroup of IET in the lattice of all subgroups of RIET defined by supernatural numbers.
- If $2^\infty | \hat{n}$ then the group $\mathrm{RIET}(\hat{n})$ is perfect, i.e. $\mathrm{RIET}(\hat{n})' = \mathrm{RIET}(\hat{n})$.
- If $2^\infty \nmid \hat{n}$ then the derived subgroup $\mathrm{RIET}(\hat{n})'$ is a proper subgroup of $\mathrm{RIET}(\hat{n})$ and consists of all the iets from $\mathrm{RIET}(\hat{n})$, which are defined by even permutations.
- $\mathrm{RIET}(2^\infty)$ is generated by the set S of all rational iets defined as:

$$S = \{(\pi_{2^n}, \sigma) \mid \sigma = (i, i+1), \ i \le 2^{n-1}\}$$

References

[1] Bier A., Sushchanskyy V., „Dense subgroups in the group of interval exchange transformations", Algebra Discrete Math. 17 (2014), p. 232 – 247

[2] Bier A., Oliynyk B., Sushchanskyy V., „Generating sets of RIET", to appear

Non-Soluble and Non-p-Soluble Length of Finite Groups

Bounding the non-p-Soluble length of finite group with conditions on their Sylow p-subgroups

Yerko Contreras Rojas
PhD student UnB
Universidade de Brasília.
Yerkocr@mat.unb.br

Abstract

Let p be a prime. Every finite group G has a normal series each of whose quotients either is p-soluble or is a direct product of nonabelian simple groups of orders divisible by p. The non-p-soluble length $\lambda_p(G)$ is defined as the number of non-p-soluble quotients in a shortest series of this kind.

We deal with the question whether, for a given prime p and a given proper group variety $\mathfrak{V}$, the non-p-soluble length $\lambda_p(G)$ of a finite group G whose Sylow p-subgroups belong to $\mathfrak{V}$ is bounded.

In joint work with Pavel Shumyatsky, we answer the question in the affirmative in some cases (working separately the case $p = 2$) for varieties of groups in which the commutators have some restrictions about their order.

Non-Soluble and Non-p-Soluble Length

Every finite group G has a normal series each of whose quotient either is soluble or is a direct product of nonabelian simple groups. In [5] the *nonsoluble length* of G, denoted by $\lambda(G)$, was defined as the minimal number of nonsoluble factors in a series of this kind: if

$$1 = G_0 \le G_1 \le \cdots \le G_{2h+1} = G$$

is a shortest normal series in which for i even the quotient G_{i+1}/G_i is soluble (possibly trivial), and for i odd the quotient G_{i+1}/G_i is a (non-empty) direct product of nonabelian simple groups, then the nonsoluble length $\lambda(G)$ is equal to h. For any prime p, a similar notion of non-p-soluble length $\lambda_p(G)$ was defined by replacing "soluble" by "p-soluble" and "simple" by "simple of order divisible by p". Recall that a finite group is said to be p-*soluble* if it has a normal series each of whose quotients is either a p-group or a p'-group. We have, $\lambda(G) = \lambda_2(G)$, since groups of odd order are soluble by the Feit–Thompson theorem [3].

We show a specific normal series that allow to obtain the non-p-soluble length of a finite group G. For this we establish some notations.

The soluble radical of a group G, the largest normal soluble subgroup, is denoted by $R(G)$. The largest normal p-soluble subgroup is called the p-*soluble radical* and it will be denoted by $R_p(G)$.

Consider the quotient $\bar{G} = G/R_p(G)$ of G by its p-soluble radical. The socle $Soc(\bar{G})$, that is, the product of all minimal normal subgroups of $\bar{G}$, is a direct product $Soc(\bar{G}) = S_1 \times \cdots \times S_m$ of nonabelian simple groups S_i of order divisible by p. Set the following series

$$1 = G_0 \le \Gamma_1 \le M_1 \le \Gamma_2 \le M_2 \cdots \le G$$

where Γ_i and M_i are defined recursively by

$$\frac{M_i}{\Gamma_{i-1}} = R_p\left(\frac{G}{\Gamma_{i-1}}\right) \qquad \frac{\Gamma_i}{M_i} = Soc\left(\frac{G}{M_i}\right).$$

The number of Γ_i appearing in this series is the non-p-soluble length of G.

Upper bounds for the nonsoluble and non-p-soluble length appear in the study of various problems on finite, residually finite, and profinite groups. For example, such bounds were implicitly obtained in the Hall–Higman paper [4] as part of their reduction of the Restricted Burnside Problem to p-groups.

The Problem

There is a long-standing problem on p-length due to Wilson (Problem 9.68 in Kourovka Notebook [1]): *for a given prime p and a given proper group variety $\mathfrak{V}$, is there a bound for the p-length of finite p-soluble groups whose Sylow p-subgroups belong to $\mathfrak{V}$?*

In [5] the following problem, analogous to Wilson's problem, was suggested.

Problem A. *For a given prime p and a given proper group variety $\mathfrak{V}$, is there a bound for the non-p-soluble length λ_p of finite groups whose Sylow p-subgroups belong to $\mathfrak{V}$?*

It was shown in [5] that an affirmative answer to Problem A would follow from an affirmative answer to Wilson's problem. On the other hand, Wilson's problem so far has seen little progress beyond the affirmative answers for soluble varieties and varieties of bounded exponent [4] (and, implicit in the Hall–Higman theorems [4], for (n-Engel)-by-(finite exponent) varieties). Problem A seems to be more tractable.

Results

In the sequel we give some useful definitions and we present some positive answer to Problem A. For instance in [5] a positive answer was obtained in the case of any variety that is a product of varieties that are either soluble or of finite exponent.

Now we define a group variety, that contains the varieties of soluble and of finite exponent groups like particular cases, and offer us some groups varieties for which Problem A has positive answer.

Definition. *Let $\mathfrak{W}(w, e)$ be the variety of all groups in which w^e-values are trivial, where w is a group-word and e is a positive integer.*

We obtain the following theorem [2]:

Theorem 1. *Let k, e be positive integers and p an odd prime. Let P be a Sylow p-subgroup of a finite group G and assume that P belongs to $\mathfrak{W}(\delta_k, p^e)$. Then $\lambda_p(G) \le k + e - 1$.*

Using the previous theorem and some others tools we obtain a generalization in the odd case of the result of Shumyatsky and Khukhro about the product of varieties

Theorem 2. *Let G be a finite group of order divisible by p, where p is an odd prime. If a Sylow p-subgroup P of G belongs to the variety $\mathfrak{W}(\delta_{k_1}, e_1)\mathfrak{W}(\delta_{k_2}, e_2)\cdots\mathfrak{W}(\delta_{k_n}, e_n)$, then $\lambda_p(G)$ is $\{k_1, e_1, ..., k_n, e_n\}$-bounded.*

We are trying to prove that the previous theorems remain valid also for $p = 2$ but so far we have not been able to prove that case. The case where $k = 0$ in Theorem 1 was handled in [5] for any prime p. Further, it is immediate from [6, Proposition 2.3] that if the order of $[x, y]$ divides 2^e for each x, y in a Sylow 2-subgroup of G, then $\lambda(G) \le e$. Hence, Theorem 1 is valid for any prime p whenever $k \le 1$.

More recently in the case $p = 2$, we obtained a new result in the way to the Theorem 2.

Given $p = 2$, the non-soluble length $\lambda(G)$ of a finite group whose Sylow 2-subgroups belong to the product of several varieties of type $\mathfrak{W}(\delta_1, e)$ is bounded.

Theorem 3. *Let G be a finite group, and let P be a Sylow 2-subgroup of G such that P belongs to a product of varieties $\mathfrak{W}(\delta_1, e_1)\mathfrak{W}(\delta_1, e_2)\cdots\mathfrak{W}(\delta_1, e_n)$. Then the non-soluble length is bounded by a function depending only of e_i, $i = 1, 2, \ldots, n$.*

The following lemma was proved in [5]. It depends on the classification of finite simple groups, and it should be noted as one of the strongest tools for obtaining the above results. We need to introduce the following definition.

Let G be a finite group and $Soc(G/R_p(G)) = S_1 \times \cdots \times S_m$. The group G induces by conjugation a permutational action on the set $\{S_1, \ldots, S_m\}$. Let $K_p(G)$ denote the kernel of this action. In [5] $K_p(G)$ was given the name of the p-*kernel subgroup* of G. Clearly, $K_p(G)$ is the full inverse image in G of $\bigcap N_{\bar{G}}(S_i)$.

Lemma. *The p-kernel subgroup $K_p(G)$ has non-p-soluble length at most 1.*

References

[1] *Unsolved problems in group theory. The Kourovka notebook*. Number 17. Russian Academy of Sciences Siberian Division, Institute of Mathematics, Novosibirsk, augmented edition, 2010.

[2] Y. Contreras-Rojas and Pavel Shumyatsky. Nonsoluble length of finite groups with commutators of small order. *Math. Proc. Cambridge Philos. Soc.*, 158(3):487–492, 2015.

[3] Walter Feit and John G. Thompson. Solvability of groups of odd order. *Pacific J. Math.*, 13:775–1029, 1963.

[4] P. Hall and Graham Higman. On the p-length of p-soluble groups and reduction theorems for Burnside's problem. *Proc. London Math. Soc. (3)*, 6:1–42, 1956.

[5] E. I. Khukhro and P. Shumyatsky. Nonsoluble and non-p-soluble length of finite groups. *Israel J. Math.*, 207(2):507–525, 2015.

[6] Pavel Shumyatsky. Commutators in residually finite groups. *Israel J. Math.*, 182:149–156, 2011.

This work was supported by CAPES and CNPq-Brazil

CAPES

Freeness and fragile words in automaton groups

Daniele D'Angeli

ISCHIA GROUP THEORY 2016

(joint work with E. Rodaro)

Introduction

Mealy automaton, or transducer:
$\mathcal{A} = \langle Q, A, \delta, \lambda \rangle$, where
$\delta : Q \times A \to Q$ is the *restriction* function and
$\lambda : Q \times A \to A$ is the *output* function. So each transition can be depicted as:

$$q \xrightarrow{a|b} p$$

to denote $\delta(q, a) = p$, and $\lambda(q, a) = b$.
These actions extends in a natural way to Q^* and A^*.

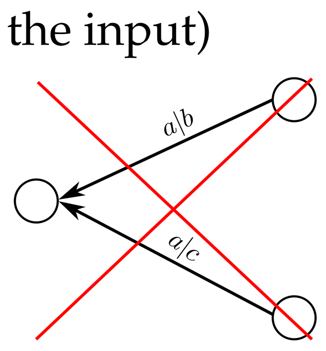
- It is *invertible* if $\lambda_q(\circ) = \lambda(q, \circ) \in Sym(A)$ for any $q \in Q$.
- Each state q gives rise to a one-to-one map $q : A^k \to A^k$.
- The inverse q^{-1} obtained building $\mathcal{A}^{-1}$ obtained from $q \xrightarrow{a|b} p$ by swapping input/output $q^{-1} \xrightarrow{b|a} p^{-1}$.
- Hence $\{q\}_{q \in Q}$ acts on A^*. These transformations give rise to a semigroup $\mathcal{S}(A) = \langle q : q \in Q \rangle$ (not invertible case), or a group $\mathcal{G}(A) = Gp\langle q : q \in Q \rangle$.

Why automaton groups?

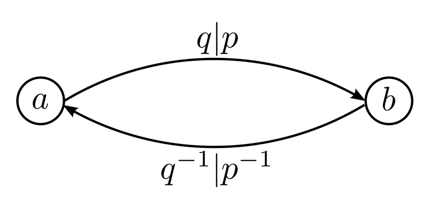
- *Burnside problem*. Infinite finitely generated torsion groups. (Grigorchuk group 1984).
- *Milnor problem*. Constructions of groups of intermediate growth. (Grigorchuk group 1984).
- *Atiyah problem*. Computation of L^2 Betti numbers. (Lamplighter group 2000. Grigorchuk, Linnell, Schick, Żuk).
- *Day problem*. New examples of amenable groups.
- *Gromov problem*. Groups without uniform exponential growth (Wilson 2002).

A geometric perspective via the enriched dual

- $\mathcal{A} = \langle Q, A, \delta, \lambda \rangle$, $\mathcal{G}(\mathcal{A}) = F_Q/N$. Is there a combinatorial description of the relations N?
- Using a "Stallings's approach" we need two ingredients: the dual of a transducer and the notion of an inverse transducer.
- The *dual* $\partial A = \langle A, Q, \lambda, \delta \rangle$
  $p \xrightarrow{a|b} q \in \mathcal{A}$ if and only if $a \xrightarrow{p|q} b \in \partial\mathcal{A}$
- In this case $\partial\mathcal{A}$ is a reversible transducer (co-deterministic in the input)



- In this case we can give a structure of *inverse transducer* (introduced by Silva)
- We *enrich* $\partial\mathcal{A} = \langle A, Q, \lambda, \delta \rangle$ with a structure of inverse transducer $\partial\mathcal{A}^{-} = \langle A, \widetilde{Q}, \lambda, \delta \rangle$ with $\widetilde{Q} = Q \cup Q^{-1}$



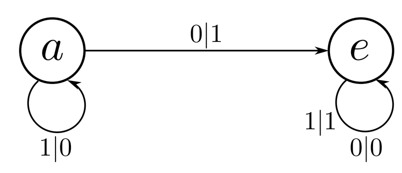
Theorem. Let $\mathcal{A} = \langle Q, A, \lambda, \delta \rangle$ be an invertible transducer, with $\mathcal{G}(\mathcal{A}) \simeq F_Q/N$. Consider the transducer $(\partial\mathcal{A})^{-} = (A, \widetilde{Q}, \circ, \cdot)$, and let

$$\mathcal{N} \subseteq \bigcap_{a \in A} L\left((\partial\mathcal{A})^{-}, a\right)$$

be the maximal subset invariant for the action of δ on $\widetilde{Q}^*$. Then $N = \overline{\mathcal{N}}$.

Example: Adding Machine

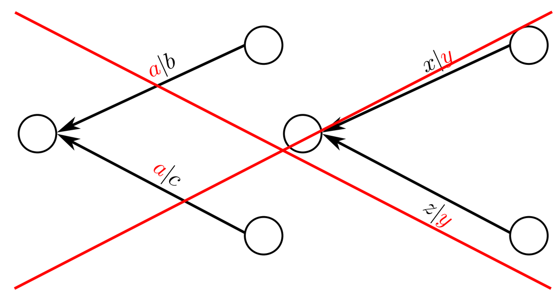
Let $\mathcal{A}$ be the following transducer:



- It performs addition of one unit in binary: $\lambda(a, 1011) = 0111$.
- The defined group is $\mathcal{G}(\mathcal{A}) \simeq \mathbb{Z}$

Free groups

- The Aleshin transducer on three state was the first example of free group $\simeq F_3$ (the original proof of the freeness was not complete, it was fixed by M. Vorobets and Y. Vorobets in 2006).
- Bounded automata cannot generate free groups (Sidki, Nekrashevych).
- Other examples of transducers defining a free group: they are all *bireversible*!
  i.e. co-deterministic in both input and output.



Some open problems regarding bireversible transducers

Virtually Free
Conjecture: If the transducer is bireversible, then $\mathcal{G}(\mathcal{A})$ is virtually free if and only if $\mathcal{G}(\partial\mathcal{A})$ is also virtually free.
Burnside
Conjecture: If the transducer is bireversible, then it does not generate an infinite Burnside group.
Freeness
Are there transducers defining free groups of rank > 1 which are not bireversible (the adding machine is not bireversible $\simeq \mathbb{Z}$)?

Freeness using the enriched dual

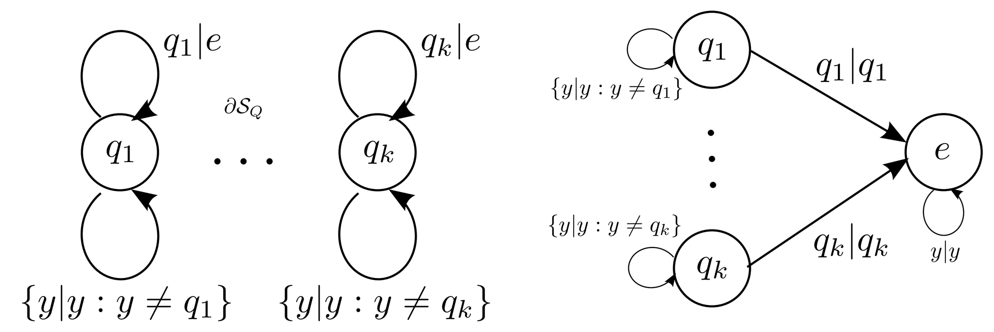
Build a transducer with a sink state which is not bireversible defining a free group.

Definition. $\mathcal{A}, \mathcal{B}$ on the same set of states Q, we say that $\mathcal{B}$ *dually embeds* into $\mathcal{A}$, in symbols $\mathcal{B} \hookrightarrow_d \mathcal{A}$, if $\partial\mathcal{B}$ is a proper connected component of $\partial\mathcal{A}$.

Corollary If $\mathcal{B} \hookrightarrow_d \mathcal{A}$ there is an epimorphism $\psi \colon \mathcal{G}(\mathcal{A}) \twoheadrightarrow \mathcal{G}(\mathcal{B})$.

A series of auxiliary transducers

Consider the following series of transducers $\partial\mathcal{S}_Q = (Q, Q \cup \{e\}, \circ, \cdot)$



Theorem. Let $\mathcal{B}$ be a transducer such that $\mathcal{G}(\mathcal{B})$ is a free group, and let $\partial A = \partial B^e \sqcup \partial\mathcal{S}_Q$. Then $\mathcal{A}$ is a transducer with sink that is accessible from any state which also defines a free group.

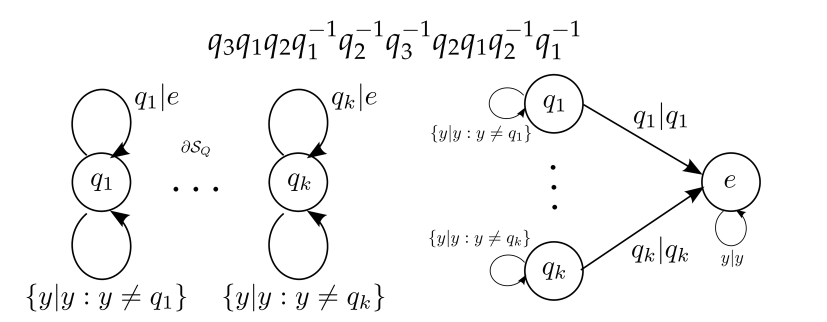
- However they do not acts transitively on A^*.

Open Problem Is there a transducer with sink (not bireversible) which acts transitively on A^* and defines a free group?

Fragile words

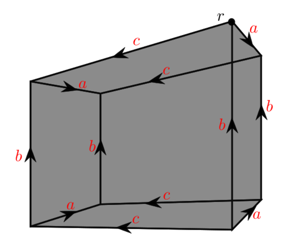
We consider the class of transducers with a sink e (acting like the identity) which is accessible from every state, the minimal reduced relations $w \in \widetilde{Q}^*$ are *fragile* in the sense that there is a letter a such that $\delta(a, w) = \epsilon$:

$$q_3 q_1 q_2 q_1^{-1} q_2^{-1} q_3^{-1} q_2 q_1 q_2^{-1} q_1^{-1}$$



- Example fragile words for $\mathcal{S}_Q$. The action of q_i is a substitutive morphism $q_i \to e$.
- Fragile words obtained by "nesting" commutators $[[a, b], c]$.
- There are other which are not express in this form: as labels of special paths in special 2-complexes for instance

$$\overline{(ab^{-1}cbc^{-1}a^{-1})(ab^{-1}a^{-1}b)(cb^{-1}aba^{-1}c^{-1})(cb^{-1}c^{-1}b)} = ab^{-1}cbc^{-1}b^{-1}a^{-1}bcb^{-1}aba^{-1}b^{-1}c^{-1}b$$



Open Problem
Characterize fragile words for the special case of $\mathcal{S}_Q$, in particular the shortest are of the commutators form.

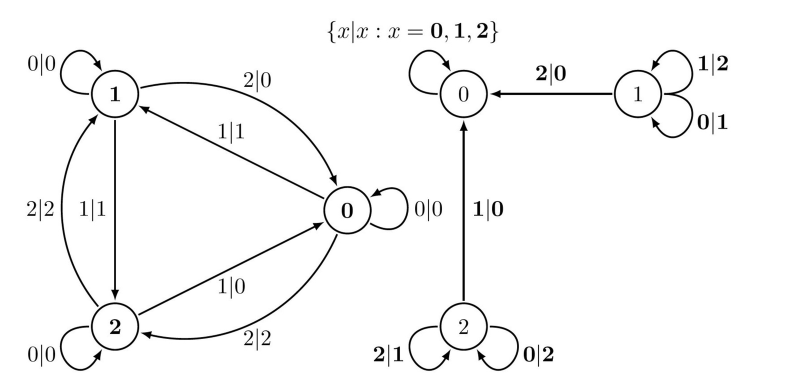
Cayley type transducers

The *0-transition Cayley machine* $\mathcal{C}(G) = (\mathcal{G}, (G), \delta, \lambda)$ is the transducer defined on the alphabet $\mathbf{G} = \{\mathbf{g} : g \in G\}$ whose transitions are of the form

- $\mathbf{g} \xrightarrow{(x)|(x)} \mathbf{gx}$ for all $g, x \in G$ such that $g \neq x$;
- $\mathbf{g} \xrightarrow{(x)|(e)} \mathbf{e}$ for all $g, x \in G$ such that $g = x$.

Similarly, we define the *bi-0-transition Cayley machine* $\widetilde{\mathcal{C}}(G) = (\mathbf{G}, (G), \delta, \lambda)$ with transitions given by:

- $\mathbf{g} \xrightarrow{(x)|(x)} \mathbf{gx}$ for all $g, x \in G$ such that $g \neq x$ and $g \neq e$;
- $\mathbf{g} \xrightarrow{(x)|(e)} \mathbf{e}$ for all $g, x \in G$ such that $g = x$ and $g \neq e$;
- $\mathbf{e} \xrightarrow{(x)|(e)} \mathbf{x}$ for all $x \in G$.



On the left the transducer $\mathcal{C}(\mathbb{Z}_3)$, on the right its dual $\partial\mathcal{C}(\mathbb{Z}_3)$.

Theorem.

- For any non trivial finite group G, the semigroup $\mathcal{S}(\partial\mathcal{C}(G))$ is free and so the group $\mathcal{G}(\partial\mathcal{C}(G))$ has exponential growth, for any non trivial group G.
- The (finite) group G is a quotient of $\mathcal{G}(\partial\mathcal{C}(G))$ and $\mathcal{G}(\partial\widetilde{\mathcal{C}}(G))$.

Open Problems The groups generated by the dual of the 0-transition Cayley machines have exponential growth. What can be said about the amenability of such groups? More generally, is it possible to find a suitable output-coloring of such transducers in order to get free groups or free products of groups? This question can be specialized for the Cayley machines, where $G = \mathbb{Z}_n$. Are the groups generated by dual of 0-transition Cayley machine $C(\mathbb{Z}_n)$ free? Are the groups generated by dual of 0-transition Cayley machine $\widetilde{C}(\mathbb{Z}_n)$ free products? In any case, does there exists a simple combinatorial description of the relations and fragile words?

Maximal subgroups of groups of intermediate growth

Alejandra Garrido (joint with Dominik Francoeur) alejandra.garrido@unige.ch

UNIVERSITÉ DE GENÈVE

Groups of intermediate growth

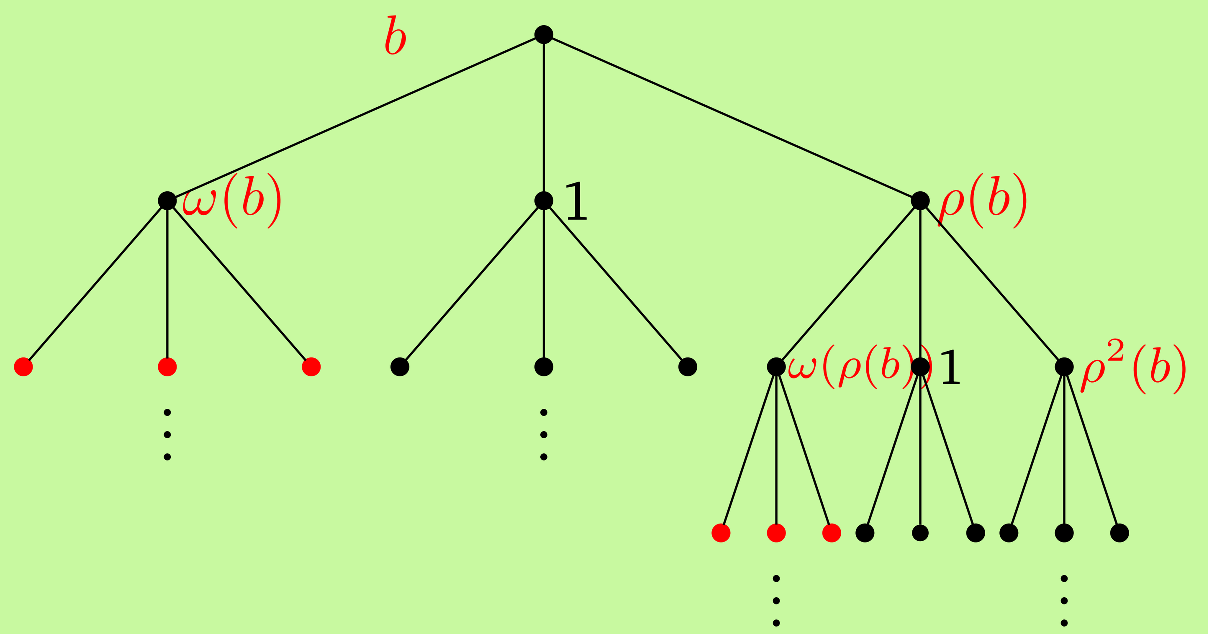
The first examples of groups of intermediate word growth were constructed by Grigorchuk in [2], who found an uncountable family of such groups. A natural generalization of these groups was defined by Šunić in [4]. The groups in question are groups of automorphisms of the rooted infinite p-regular tree for any prime p.

Definition (Šunić groups [4])**.** Let $A := \langle a \rangle \cong \mathbb{Z}/p\mathbb{Z}$ and $B \cong A^m$ for some $m \geq 1$. For an epimorphism $\omega : B \to A$ and an automorphism $\rho : B \to B$, define

$$G_{\omega,\rho} = \langle a, B \rangle$$

where a acts as the p-cycle $(1, 2, \ldots, p)$ on the first level of the tree and the action of $b \in B$ is recursively defined by

$$b := (\omega(b), 1, 1, \ldots, \rho(b)).$$



Examples

- "First" Grigorchuk group: $p = m = 2$ and $b = (a, c), c = (a, d), d = (1, d)$ (torsion)
- Grigorchuk–Erschler group: $p = m = 2$ and $b = (a, b), c = (a, d), d = (1, c)$ (non-torsion)
- Fabrykowski–Gupta group: $p = 3, m = 1$ and $b = (a, 1, b)$ (non-torsion).

Properties

- All are self-similar.
- Except for the infinite dihedral group ($p = 2, m = 1$): branch groups, intermediate growth (thus amenable but not elementary amenable).
- Some of them are torsion groups, some not.

Maximal subgroups

In a finitely generated group, every subgroup is contained in a maximal subgroup, so it is natural to study the maximal subgroups of a given group. Maximal subgroups also correspond to primitive actions of the ambient group (primitive actions are the building blocks of all other actions).

When does a group have all maximal subgroups of finite index (i.e., all primitive actions are of finite degree)?

Linear groups Margulis and Soifer, 1981: A finitely generated linear group has all maximal subgroups of finite index if and only if it is virtually solvable.

Branch groups Pervova [3]: For each torsion group in Grigorchuk's family, all maximal subgroups have finite index. Same for torsion GGS groups. Same result by Alexoudas–Klopsch–Thillaisundaram for torsion groups in a generalized family of GGS groups.

Bondarenko [1]: There exist finitely generated branch groups with maximal subgroups of infinite index. These branch groups are subgroups of iterated wreath products of finite perfect groups and are perfect themselves. Result also holds for similar groups constructed by P.M. Neumann (1986), D. Segal (2001) and J.S. Wilson (2002) (groups of non-uniform exponential growth).

Question

What about the non-torsion groups in Grigorchuk and Šunić's families? They have intermediate growth. Are their maximal subgroups of finite index?

References

[1] I. V. Bondarenko. Finite generation of iterated wreath products, Arch. Math (2010) 95:301–308.
[2] R. I. Grigorchuk. Degrees of growth of finitely generated groups, and the theory of invariant means, Math. USSR Izv. (1985) 25:259–300.
[3] E. Pervova. Profinite completions of some groups acting on trees, J. Algebra (2007) 310:858–879.
[4] Z. Šunić. Hausdorff dimension in a family of self-similar groups, Geom. Ded. (2007) 124:213–236.

First theorem: the profinite and $\operatorname{Aut} T$ topologies

Dense subgroups in the profinite topology A group G has a maximal subgroup of infinite index if and only if it has a proper subgroup H which is dense in the profinite topology, i.e., $HN = G$ for every $N \trianglelefteq G$ of finite index.

Theorem 1. *Every group G in Šunić's family has the* ***congruence subgroup property****: every $N \trianglelefteq G$ of finite index contains some level stabilizer $\mathrm{St}_G(n)$. In particular, G has a maximal subgroup of infinite index if and only if it has a proper subgroup H such that $H\,\mathrm{St}_G(n) = G$ for every n.*

Main theorem: maximal subgroups of infinite index

Theorem 2. *Let $G = \langle a, B \rangle$ be a non-torsion Šunić group acting on the binary tree and pick $b \in B$ such that ab has infinite order. If q is an odd prime, then $H_q = \langle (ab)^q, B \rangle < G$ is a proper subgroup, dense in the profinite topology. Hence G contains infinitely many maximal subgroups of infinite index.*

Third theorem: maximal subgroups of finite index

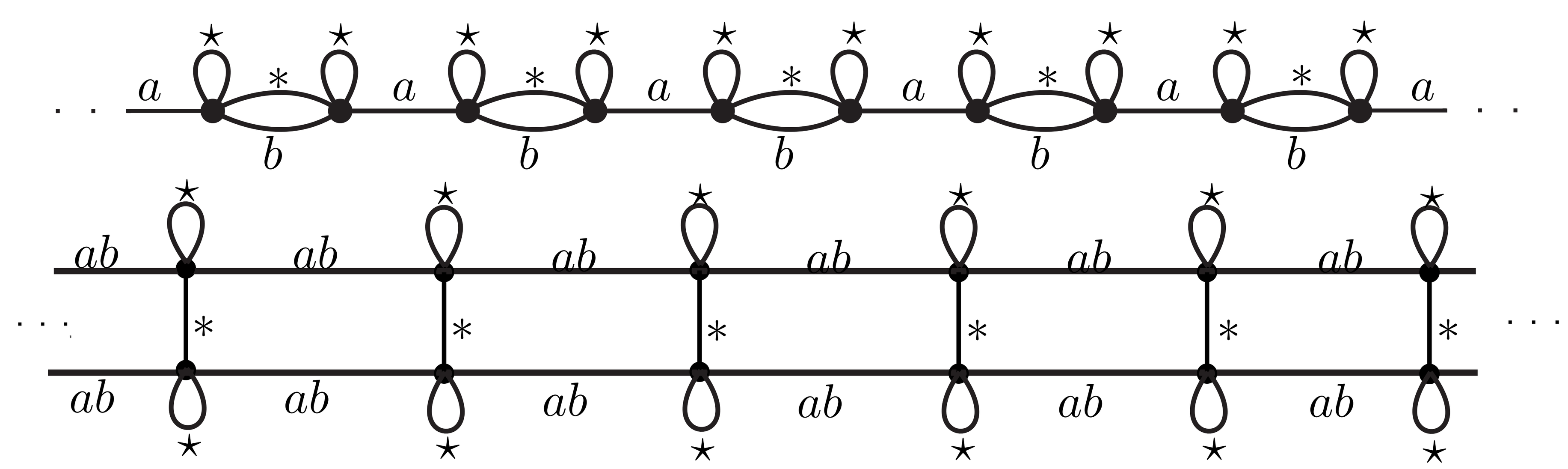
Theorem 3. *The Fabrykowski–Gupta group acting on the ternary tree has all maximal subgroups of finite index.*

Proof of main theorem

First part: H_q is dense

Proposition (P.-H. Leemann)**.** *Let T be the d-regular infinite rooted tree and let $G = \langle g_1, g_2, \ldots \rangle \leq \operatorname{Aut} T$ be countably generated. Then for any $m_1, m_2, \ldots$ coprime with $d!$ the subgroup $H = \langle g_1^{m_1}, g_2^{m_2}, \ldots \rangle$ satisfies $H\,\mathrm{St}_G(n) = G$ for every n.*
If G has the congruence subgroup property, then H is dense in the profinite topology: $HN = G$ for every $N \trianglelefteq G$ of finite index.

Second part: H_q is proper To show that $ab \notin H_q$, we examine the Schreier graphs (orbital graphs) of G on the boundary of the tree. They are all either one- or two-ended. It suffices to consider the two-ended ones. The figure below shows a two-ended Schreier graph with respect to the generating set $\{a, B\}$ and then the same graph with respect to $\{ab, B\}$. The symbol $*$ denotes elements of B which act non-trivially while $\star$ denotes those that act trivially (there may be multiple such elements in each case).



Suppose that $w = ab$ for some $w \in H_q$. Then w and ab should produce the same paths starting at any vertex in the Schreier graph shown above. But $(ab)^{\pm q}$ moves q edges to the left or right while each $*$ moves vertically, preserving the "horizontal coordinate", and $\star$ does nothing. So no word in $\{(ab)^q, B\}$ can act like ab on this orbit of a boundary point.

METU

Beauville structures in p-central quotients

Şükran Gül

Department of Mathematics, Middle East Technical University

gsukran@metu.edu.tr

Introduction

A *Beauville surface* of unmixed type is a compact complex surface isomorphic to $(C_1 \times C_2)/G$, where C_1 and C_2 are algebraic curves of genus at least 2 and G is a finite group acting freely on $C_1 \times C_2$ and faithfully on the factors C_i such that $C_i/G \cong \mathbb{P}_1(\mathbb{C})$ and the covering map $C_i \to C_i/G$ is ramified over three points for $i = 1, 2$. Then the group G is said to be a *Beauville group*. For a couple of elements $x, y \in G$, we define

$$\sum(x,y) = \bigcup_{g\in G} \left(\langle x\rangle^g \cup \langle y\rangle^g \cup \langle xy\rangle^g\right),$$

that is, the union of all subgroups of G which are conjugate to $\langle x\rangle$, to $\langle y\rangle$ or to $\langle xy\rangle$. Then G is a Beauville group if and only if the following conditions hold:

1. G is a 2-generator group.
2. There exists a pair of generating sets $\{x_1, y_1\}$ and $\{x_2, y_2\}$ of G such that $\Sigma(x_1, y_1) \cap \Sigma(x_2, y_2) = 1$.

Then $\{x_1, y_1\}$ and $\{x_2, y_2\}$ are said to form a *Beauville structure* for G.

In [1], it has been shown that there are infinitely many Beauville p-groups for $p \geq 5$. The existence of infinitely many Beauville 3-groups is proved in [4]; however, the proof does not yield explicit groups. The first explicit infinite family of Beauville 3-groups has been recently given in [3].

In [2], Boston conjectured that if $p \geq 5$ and F is either the free group on two generators or the free product of two cyclic groups of order p, then its p-central quotients $F/\lambda_n(F)$ are Beauville groups. We prove Boston's conjecture. In fact, in the case of the free product, we extend the result to $p = 3$.

The free group on two generators [5]

Definition

For any group G, the normal series

$$G = \lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G) \geq \ldots$$

given by $\lambda_n(G) = [\lambda_{n-1}(G), G]\lambda_{n-1}(G)^p$ for $n > 1$ is called the *p-central series* of G. A quotient group $G/\lambda_n(G)$ is said to be a *p-central quotient* of G.

Lemma 1

Let $F = \langle x, y\rangle$ be the free group on two generators. Then $x^{p^{n-2}}$ and $y^{p^{n-2}}$ are linearly independent modulo $\lambda_n(F)$ for $n \geq 2$.

Lemma 2

If $G = F/\lambda_n(F)$, the power subgroups $M^{p^{n-2}}$ are all different and of order p in $\lambda_{n-1}(F)/\lambda_n(F)$, as M runs over the $p+1$ maximal subgroups of G. In particular, all elements in $M \smallsetminus \Phi(G)$ are of order p^{n-1}.

Theorem 1

Let $F = \langle x, y\rangle$ be the free group on two generators. Then a p-central quotient $F/\lambda_n(F)$ is a Beauville group if and only if $p \geq 5$ and $n \geq 2$.

The free product of two cyclic groups of order p [5]

Theorem 2

Let $F = \langle x, y \mid x^p, y^p\rangle$ be the free product of two cyclic groups of order p. Then a p-central quotient $F/\lambda_n(F)$ is a Beauville group if and only if $p \geq 5$ and $n \geq 2$ or $p = 3$ and $n \geq 4$.

Thus for $p = 3$, p-central quotients in Theorem 2 constitute an infinite family of Beauville 3-groups.

Recall that

- The *Nottingham group* $\mathcal{N}$ over the field $\mathbb{F}_p$, for odd p, is the (topological) group of normalized automorphisms of the ring $\mathbb{F}_p[[t]]$ of formal power series.
- For any positive integer k, the automorphisms $f \in \mathcal{N}$ such that $f(t) = t + \sum_{i\geq k+1} a_i t^i$ form an open normal subgroup $\mathcal{N}_k$ of $\mathcal{N}$ of p-power index.

In [3], it has been shown that if $p = 3$ then a quotient $\mathcal{N}/\mathcal{N}_k$ is a Beauville group if and only if $k \geq 6$ and $k \neq z_m$ for $m \geq 1$, where $z_m = p^m + p^{m-1} + \cdots + p + 2$.

Theorem 3

A quotient group $F/\lambda_n(F)$ is not isomorphic to any of $\mathcal{N}/\mathcal{N}_k$ for $n > 4$. On the other hand, $F/\lambda_4(F)$ is isomorphic to $\mathcal{N}/\gamma_4(\mathcal{N})$.

As a consequence of Theorem 3, the infinite family of Beauville 3-groups in Theorem 2 only coincides at the group of order 3^5 with the explicit infinite family of Beauville 3-groups in [3].

References and Acknowledgments

[1] N. Barker, N. Boston, and B. Fairbairn, A note on Beauville p-groups, *Experiment. Math.* **21** (2012), 298–306.

[2] N. Boston, A survey of Beauville p-groups, in Beauville Surfaces and Groups, editors I. Bauer, S. Garion, A. Vdovina, *Springer Proceedings in Mathematics & Statistics*, Volume 123, Springer, 2015, pp. 35–40.

[3] G.A. Fernández-Alcober and Ş. Gül, Beauville structures in finite p-groups, preprint, available at `arXiv:1507.02942 [math.GR]`.

[4] G. González-Diez and A. Jaikin-Zapirain, The absolute Galois group acts faithfully on regular dessins and on Beauville surfaces, *Proc. London Math. Soc* (3) **111** (2015), 775–796.

[5] Ş. Gül, Beauville structures in p-central quotients, preprint.

I would like to thank G. Fernández-Alcober, N. Gavioli and C. M. Scoppola for helpful comments and suggestions.

Maximal Subgroup Containment in Direct Products

Dr. Dandrielle Lewis
University of Wisconsin-Eau Claire

Purpose

Using the main theorem from [2] that characterizes containment of subgroups in a direct product, we provide a characterization of maximal subgroups contained in a direct product. We also provide an example of our main theorem to a maximal subgroup in $A_4 \times A_4$.

History

In [2], we specify exact conditions on two subgroups U_1 and U_2 of the direct product of two groups A and B that characterize when $U_2 \leq U_1$. As applications, we calculated and presented the subgroup lattice of $Q_8 \times Q_8$, where Q_8 is the quaternion group of order 8. Other examples provided in Dandrielle Lewis's dissertation include groups that are supersolvable, even nilpotent, which made order of subgroup sufficient for determining maximality of one subgroup in another.

The article [4] tackled finding the maximal subgroups of a direct product, but to apply the ideas of [2] to get the subgroup lattice of a non-supersolvable group we want a characterization of the maximal subgroups of a subgroup of a direct product. That is what the maximal subgroup containment characterization accomplishes.

Goursat's Theorem

Theorem 1 [1]

Let **A** *and* **B** *be groups. Then there exists a bijection between the set of all subgroups of* $\mathbf{A} \times \mathbf{B}$ *and the set of all triples* $\left(\frac{I}{J}, \frac{L}{K}, \sigma\right)$, *where* $\frac{\mathbf{I}}{\mathbf{J}}$ *is a section of* **A**, $\frac{\mathbf{L}}{\mathbf{K}}$ *is a section of* **B**, *and* $\sigma : \frac{I}{J} \to \frac{L}{K}$ *is an isomorphism between the sections.*

Notation

Consider the projections

$$\pi_A : A \times B \to A$$

and

$$\pi_B : A \times B \to B,$$

and let $U \leq A \times B$ correspond to the triple $\left(\frac{I}{J}, \frac{L}{K}, \sigma\right)$. It follows that:

$$U \cap A \lhd \pi_A(U), U \cap B \lhd \pi_B(U),$$

and

$$\sigma : \frac{\pi_A(U)}{U \cap A} \to \frac{\pi_B(U)}{U \cap B}$$

is an isomorphism.

- Now, let $I = \pi_A(U)$, $J = U \cap A$, $L = \pi_B(U)$, and $K = U \cap B$.
- The subgroup structure given by Goursat's Theorem is $U = \{(a, b) \mid a \in I, b \in L, \text{and } (aJ)^\sigma = bK\}$.

Containment Theorem

Theorem 2 [2]

Suppose $U_2, U_1 \leq A \times B$, where U_1 is given by the triple $\left(\frac{I_1}{J_1}, \frac{L_1}{K_1}, \sigma_1\right)$ and U_2 is given by the triple $\left(\frac{I_2}{J_2}, \frac{L_2}{K_2}, \sigma_2\right)$. Then $U_2 \leq U_1$ if and only if:

1. $I_2 \leq I_1$, $J_2 \leq J_1$, $L_2 \leq L_1$, **and** $K_2 \leq K_1$

2. $\left(\frac{I_2 J_1}{J_1}\right)^{\sigma_1} = \frac{L_2 K_1}{K_1}$

3. $\left(\frac{I_2 \cap J_1}{J_2}\right)^{\sigma_2} = \frac{L_2 \cap K_1}{K_2}$

4.

$$\begin{tikzcd}
\frac{I_2 J_1}{J_1} \arrow[r, "\widetilde{\sigma}_1"'] \arrow[d, "\theta_1"] & \frac{L_2 K_1}{K_1} \arrow[d, "\theta_2"'] \\
\frac{I_2}{I_2 \cap J_1} \arrow[r, "\widetilde{\sigma}_2"] & \frac{L_2}{L_2 \cap K_1}
\end{tikzcd}$$

Main Theorem: Maximal Subgroup Containment Theorem

Theorem 3 [3]

Suppose $U_n \leq A \times B$ with U_n corresponding to the triple $\left(\frac{I_n}{J_n}, \frac{L_n}{K_n}, \sigma_n\right)$, where $n = 1, 2$.

Then $U_2 < \cdot U_1$ if and only if

1. $U_2 \leq U_1$, and

2. If

(I.) $J_1 \times K_1 \leq U_2$, then $I_2 < \cdot I_1$.

(II.) $J_1 \times K_1 \not\leq U_2$, then either

(a) $K_1 \leq U_2$ and consequently $I_2 < \cdot I_1$ and $L_2 = L_1$, or

(b) $J_1 \leq U_2$ and consequently $L_2 < \cdot L_1$ and $I_2 = I_1$, or

(c) $J_1 \not\leq U_2$ and $K_1 \not\leq U_2$ and consequently $I_2 = I_1$, $L_2 = L_1$, and $\frac{J_1}{J_2}$ is a chief factor of I_1.

Properties of $A_4 \times A_4$ and Notation

Consider the direct product $A_4 \times A_4$.

There are 216 subgroups of $A_4 \times A_4$: the trivial subgroup and the group itself, 12 subgroups of order 2, 43 subgroups of order 3, 35 subgroups of order 4, 24 subgroups of order 6, 15 subgroups of order 8, 16 subgroups of order 9, 50 subgroups of order 12, 1 subgroup of order 16, 6 subgroups of order 24, 8 subgroups of order 36, and 4 subgroups of order 48.

Notation for subgroups of A_4:

- Denote the Klein 4-group as V, and
- The four cyclic groups of order 3 as $F_i =< f_i >$, where $1 \leq i \leq 4$

Example/Application of Main Theorem (Theorem 3)

Let U_1 be the subgroup of order 48, in $A_4 \times A_4$, corresponding to the triple $\left(\frac{A_4}{V}, \frac{A_4}{V}, id\right)$.

- $U_1 < \cdot A_4 \times A_4$ by Theorem 3 (II.)(c).
- To determine the maximal subgroups, U_2, contained in U_1, we need to verify (i) and (ii) from Theorem 3.

  Verifying (i) for U_2 is routine. So, let's verify (ii).
- For U_1, observe that $J_1 = K_1 = V$, and $I_1 = L_1 = A_4$.
- If $V \times V \leq U_2$ and $I_2 < \cdot A_4$, then $I_2 = V$.

  So, (ii)(I.) gives one maximal subgroup that corresponds to the triple $\left(\frac{V}{V}, \frac{V}{V}, id\right)$, which is the direct product $V \times V$.
- If $V \times V \not\leq U_2$, $V \leq U_2$, $I_2 < \cdot A_4$ and $L_2 = A_4$, then $I_2 = F_i$ and $K_2 = V$.

  So, (ii)(II.)(a) gives 4 maximal subgroups that correspond to the triples $\left(\frac{F_i}{1}, \frac{A_4}{V}, id\right)$, which are $(1 \times V) < (f_i, f_i) >$.
- Analogously, with respect to factors, (ii)(II.)(b) gives 4 maximal subgroups that correspond to the triples $\left(\frac{A_4}{V}, \frac{F_i}{1}, id\right)$, which is $(V \times 1) < (f_i, f_i) >$.
- If $V \times V \not\leq U_2$, $J_1 = K_1 = V \not\leq U_2$, $I_2 = A_4$, $L_2 = A_4$, and $\frac{V}{J_2}$ is an A_4 chief factor, then $J_2 = 1 = K_2$.

  So, (ii)(II.)(c) gives 12 maximal subgroups that correspond to the triples $\left(\frac{A_4}{1}, \frac{A_4}{1}, \tau_a\right)$, where τ_a, $a \in A_4$, is the inner automorphism induced by a.
  More specifically, these subgroups are diagonal subgroups of $A_4 \times A_4$.
  In order to have set containment, a must be an even permutation.

Therefore, (ii) is satisfied, and by Theorem 3, U_1 contains 21 maximal subgroups, including 1 of order 16 and 20 of order 12.

References

[1] Goursat, E. (1889). Sur les substitutions orthogonales et les divisions règuliéres de l'espace. *Ann. Sci. Ècole Norm. Sup.* **6** p. 9-102.

[2] Lewis, D. & Brewster, B. (2011). A Characterization Of Subgroup Containment In Direct Products. *Ricerche di Matematica*. **61**(2) p. 347-354.

[3] Lewis, D. & Brewster, B. (2016). Maximal Subgroup Containment in Direct Products *Advances in Group Theory and Applications* **1** p. 131-137.

[4] Thévenaz, J., Maximal Subgroups of Direct Products, *Journal of Algebra*, 1997, Volume 198(2), Pages 352-361.

On the dimension of the product $[L_2, L_2, L_1]$ in free Lie algebras

Nil Mansuroğlu
Ahi Evran University

1. Introduction

Let L be the free Lie algebra of rank $r \geq 2$ over a field K on $X = \{x_1, x_2, \ldots, x_r\}$ which is a set ordered by $x_1 < x_2 < \ldots < x_r$. The free centre-by-metabelian Lie algebra G on X is defined as the quotient

$$G = L/[L'', L],$$

where L'' is the second derived ideal of L. The second derived ideal G'' of G is defined to be quotient

$$G'' = L''/[L'', L].$$

The free centre-by-metabelian Lie algebra G has a natural grading by degree. We let G_n denote the degree n homogeneous component of G, that is spanned by Lie products of degree n in the free generators of G, and write G''_n for the degree n homogeneous component of the second derived ideal:

$$G''_n = G'' \cap G_n.$$

We let G_q denote the fine homogeneous component of G of multidegree q for a fixed composition $q = (q_1, q_2, \ldots, q_r)$ of n, that is the submodule of G generated by all Lie products of partial degree q_i with respect to x_i for $1 \leq i \leq r$. Each of the homogeneous components G_n can be written as a direct sum of fine homogeneous components,

$$G_n = \bigoplus_{q \models n} G_q.$$

If all non-zero parts of q are equal to 1, a fine homogeneous component G_q of multidegree q is called multilinear. We define Kuz'min elements as Lie monomials of the form

$$[[y_1, y_2], [y_3, y_4, y_5, \ldots, y_n]]$$

for all $y_i \in X$ with $i \in \{1, 2, \dots, n\}$ such that

$$y_1 > y_2,\ y_3 > y_4,\ y_1 \geq y_3,\ y_4 \leq y_2 \leq y_5 \leq \dots \leq y_n$$

and t-elements are defined as

$$\begin{aligned} w(y_1, y_2, y_3, y_4; y_5 \dots y_n) &= [[y_1, y_2], [y_3, y_4, y_5, \dots, y_n]] \\ &+ [[y_2, y_3], [y_1, y_4, y_5, \dots, y_n]] + [[y_3, y_1], [y_2, y_4, y_5, \dots, y_n]] \end{aligned}$$

for all $y_i \in X$ with $i = 1, 2, \dots, n$.

2. Main Results

Theorem 1. *Let G be the free centre-by-metabelian Lie algebra of rank $r > 1$ over a field K of characteristic other than 2. Then the dimensions of the homogeneous components and the fine homogeneous components of the second derived algebra G'' are as follows:*

(i) If $n \geq 5$ is odd, then

$$\dim(G''_n) = \frac{1}{2} r(n-3) \binom{n+r-3}{n-1}.$$

Moreover, if $q \models n$ is a composition of n in r parts such that k of the parts are non-zero and m of the parts are 1, then

$$\dim(G''_q) = \binom{k}{2} - m.$$

(ii) If $n \geq 6$ is even, then

$$\dim(G''_n) = \binom{n-1}{2} \binom{n+r-3}{n}.$$

Moreover, if $q \models n$ is a composition of n in r parts such that k of the parts are non-zero, then

$$\dim(G''_q) = \binom{k-1}{2}.$$

Let $q \models 5$ be a composition of 5 in r parts such that k of the parts are non-zero and m of the parts are 1. The homogeneous component of G''_5 is the sum of the fine homogeneous components G''_q, namely,

$$G''_5 = \bigoplus_{q \models 5} G''_q.$$

Lemma 1. *Over any field K, let G''_5 be the degree 5 homogeneous component of the second derived ideal G''. Then*

$$\dim[L_2, L_2, L_1] = \dim[L_3, L_2] - \dim G''_5.$$

Proof. Recall that the free centre-by-metabelian Lie algebra G is the quotient $L/[L'', L]$, where L'' is the second derived ideal of L. Then G is a graded algebra, and we denote its degree n homogeneous component by G_n. Here $G_n \cong L_n/(L_n \cap [L'', L])$. Moreover, the second derived ideal of G is the quotient $G'' = L''/[L'', L]$. As we have known, $G''_n = G'' \cap G_n$. We are interested in $G_5 \cap G''$.

The second derived ideal of L can be expressed as $[L_2, L_2] \oplus [L_3, L_2] \oplus ([L_4, L_2] + [L_3, L_3]) \oplus \ldots$. Hence, we have

$$\begin{aligned}[L'', L] =& [[L_2, L_2] \oplus [L_3, L_2] \oplus \ldots, L_1 \oplus L_2 \oplus \ldots] \\ =& [L_2, L_2, L_1] \oplus [L_3, L_2, L_1] \oplus \ldots.\end{aligned}$$

For degree 5, we have

$$\begin{aligned}G''_5 =& G_5 \cap G'' \\ \cong& (L_5/(L_5 \cap [L'', L]) \cap L''/[L'', L] \\ \cong& (L_5 \cap L'')/(L_5 \cap [L'', L]).\end{aligned}$$

Since L'' has only the subspace $[L_3, L_2]$ and $[L'', L]$ has only the subspace $[L_2, L_2, L_1]$ for degree 5, we have $L_5 \cap L'' = [L_3, L_2]$ and $L_5 \cap [L'', L] = [L_2, L_2, L_1]$. Hence,

$$G''_5 \cong [L_3, L_2]/[L_2, L_2, L_1].$$

As a result, we obtain

$$\dim G_5'' = \dim[L_3, L_2] - \dim[L_2, L_2, L_1]$$

or

$$\dim[L_2, L_2, L_1] = \dim[L_3, L_2] - \dim G_5''.$$

This completes the proof of the lemma.

□

Theorem 2. *Let L be the free Lie algebra of rank r over a field K. If $r \geq 5$, then the dimension of $[L_2, L_2, L_1]$ over a field of characteristic 2 is strictly less than the dimension of $[L_2, L_2, L_1]$ over a field of characteristic other than 2.*

Proof. Let $q \models 5$ be a composition of 5 in r parts such that k of the parts are non-zero and m of the parts are 1. The homogeneous component of G_5'' is the sum of the fine homogeneous components G_q'', namely,

$$G_5'' = \bigoplus_{q \models 5} G_q''.$$

Suppose that K is the field of characteristic other than 2. According to Theorem 1, we have

$$\dim(G_q'') = \binom{k}{2} - m.$$

If q is multilinear, namely, $m = k$,

$$\dim(G_q'') = \binom{k}{2} - k = \frac{1}{2}k(k-1) - k = \binom{k-1}{2} - 1.$$

Suppose that Char$K$=2. According to Theorem 1, if q is multilinear, then

$$\dim(G_q'') = \binom{k-1}{2}.$$

If at least one of the parts of q is greater than 1, then

$$\dim(G_q'') = \binom{k}{2} - m.$$

We can show the formulae of dimensions for G''_q in the following diagram:

	$\mathrm{Char}K = 2$	$\mathrm{Char}K \neq 2$
q multilinear	$\binom{k-1}{2}$	$\binom{k-1}{2} - \mathbf{1}$
q non-multilinear	$\binom{k}{2} - m$	$\binom{k}{2} - m$

By this diagram, it is easy to see that for q multilinear composition of 5, the dimension of G''_q over a field of characteristic 2 is more by 1 than the dimension of G''_q over a field of characteristic other than 2. Therefore, since the dimension of G''_5 is the sum of the dimensions of the fine homogeneous components G''_q, the dimension of G''_5 over a field of characteristic 2 is greater than the dimension of G''_q over a field of characteristic other than 2.

By Lemma 1, we have

$$\dim[L_2, L_2, L_1] = \dim[L_3, L_2] - \dim G''_5.$$

Therefore, it is clear to see that the dimension of $[L_2, L_2, L_1]$ over a field of characteristic 2 is strictly less than the dimension of $[L_2, L_2, L_1]$ over a field of characteristic other than 2.

$\square$

3. References

[**1**] Kuz'min, Yu.V., Free center-by-metabelian groups, Lie algebras, and D-groups, *Math. USSR lzv.* 11 1, (1977).

[**2**] Mansuroğlu, N., Products of homogeneous subspaces in free Lie algebra, MSc thesis, Universitey of Manchester, (2010).

[**3**] Mansuroğlu, N., The structure of second derived ideal in free centre-by-metabelian Lie rings, PhD thesis, Universitey of Manchester, (2014).

[**4**] Mansuroğlu, N.; Stöhr, R., On the dimension of products of homogeneous subspaces in free Lie algebras. *Internat. J. Algebra Comput.* 23 (2013), no. 1, 205–213. MR3040808

[**5**] Mansuroğlu, N.; Stöhr, R., Free centre-by-metabelian Lie rings, *Quart.J.Math.*, Advance access published May 17 2013, doi:10.1093/qmath/hat017, (2013).

Normalizers of Primitive Permutation Groups

Robert M. Guralnick*, Attila Maróti, László Pyber****

*University of Southern California, Los Angeles, USA and **MTA Alfréd Rényi Institute of Mathematics, Budapest, Hungary

guralnic@usc.edu, maroti.attila@renyi.mta.hu, pyber.laszlo@renyi.mta.hu

Abstract

Let G be a transitive normal subgroup of a permutation group A of finite degree n. The factor group A/G can be considered as a certain Galois group and one would like to bound its size. One of the results is that $|A/G| < n$ if G is primitive unless $n = 3^4, 5^4, 3^8, 5^8$, or 3^{16}. This bound is sharp when n is prime. In fact, when G is primitive, $|\mathrm{Out}(G)| < n$ unless G is a member of a given infinite sequence of primitive groups and n is different from the previously listed integers. Many other results of this flavor are established not only for permutation groups but also for linear groups and Galois groups.

1 Introduction

Aschbacher and the first author showed [AG2] that if A is a finite permutation group of degree n and A' is its commutator subgroup, then $|A : A'| \leq 3^{n/3}$, furthermore if A is primitive, then $|A : A'| \leq n$. These results were motivated by a problem in Galois theory. For another motivation we need a definition. Let $\mathcal{N}$ be a normal series for a finite group X such that every quotient in $\mathcal{N}$ either involves only noncentral chief factors or is an elementary abelian group with at least one central chief factor. Define $\mu(\mathcal{N})$ to be the product of the exponents of the quotients which involve central chief factors. Let $\mu(X)$ be the minimum of the $\mu(\mathcal{N})$ for all possible choices of $\mathcal{N}$. This invariant is an upper bound for the exponent of X/X'. In [G2] it was shown that if A is a permutation group of degree n, then $\mu(A) \leq 3^{n/3}$, furthermore if A is transitive, then $\mu(A) \leq n$, and if A is primitive with $A'' \neq 1$, then the exponent of A/A' is at most $2 \cdot n^{1/2}$. These results were also motivated by Galois theory. We prove similar statements and obtain corresponding results in Galois theory.

Let G be a normal subgroup of a permutation group A of finite degree n. The factor group A/G is studied. It is often assumed that G is transitive (this is very natural from the point of view of Galois groups and the results are much weaker without this assumption). Our first result is the following.

Theorem 1.1. *Let G and A be permutation groups of finite degree n with $G \lhd A$. Suppose that G is primitive. Then $|A/G| < n$ unless G is an affine primitive permutation group and the pair $(n, A/G)$ is* $(3^4, \mathrm{O}_4^-(2))$, $(5^4, \mathrm{Sp}_4(2))$, $(3^8, \mathrm{O}_6^-(2))$, $(3^8, \mathrm{SO}_6^-(2))$, $(3^8, \mathrm{O}_6^+(2))$, $(3^8, \mathrm{SO}_6^+(2))$, $(5^8, \mathrm{Sp}_6(2))$, $(3^{16}, \mathrm{O}_8^-(2))$, $(3^{16}, \mathrm{SO}_8^-(2))$, $(3^{16}, \mathrm{O}_8^+(2))$, *or* $(3^{16}, \mathrm{SO}_8^+(2))$. *Moreover if A/G is not a section of $\Gamma\mathrm{L}_1(q)$ when $n = q$ is a prime power, then $|A/G| < n^{1/2} \log_2 n$ for $n \geq 2^{14000}$.*

The $n-1$ bound in Theorem 1.1 is sharp when n is prime and G is a cyclic group of order n. For more information about the eleven exceptions in Theorem 1.1 and for a few other examples see the paper. Note that for every prime p there are infinitely many primes r such that the primitive permutation group $G \leq \mathrm{A}\Gamma\mathrm{L}_1(q)$ of order $np = qp = r^{p-1}p$ satisfies $|N_{\mathrm{S}_n}(G)/G| = (n-1)(p-1)/p$. It will also be clear from our proofs that the bound $n^{1/2} \log_2 n$ in Theorem 1.1 is asymptotically sharp apart from a constant factor at least $\log_9 8$ and at most 1.

Our second result concerns the size of the outer automorphism group $\mathrm{Out}(G)$ of a primitive subgroup G of the finite symmetric group S_n.

Theorem 1.2. *Let $G \leq \mathrm{S}_n$ be a primitive permutation group. Then $|\mathrm{Out}(G)| < n$ unless $|\mathrm{Out}(G)| = |N_{\mathrm{S}_n}(G)/G| \geq n$ (see Theorem 1.1 for the seven exceptions) or $n = q^2$ with $q = 2^e$, $e > 1$, and $G = (C_2)^{2e} : \mathrm{L}_2(q)$.*

Note that if G is a member of the infinite sequence of exceptions in Theorem 1.2, then $|\mathrm{Out}(G)| < (n \log_2 n)/2$.

Next we state an asymptotic version of Theorem 1.2. For this we need a definition. Let $\mathcal{C}$ be the class of all affine primitive permutation groups G with an almost simple point-stabilizer H with the property that the socle $\mathrm{Soc}(H)$ of H acts irreducibly on the socle of G and $\mathrm{Soc}(H)$ is isomorphic to a finite simple classical group such that its natural module has dimension at most 6.

Theorem 1.3. *Let $G \leq \mathrm{S}_n$ be a primitive permutation group. Suppose that if $n = q$ is a prime power then G is not a subgroup of $\mathrm{A}\Gamma\mathrm{L}_1(q)$. If G is not a member of the infinite sequence of examples in Theorem 1.2, then $|\mathrm{Out}(G)| < 2 \cdot n^{3/4}$ for $n \geq 2^{14000}$. Moreover if G is not a member of $\mathcal{C}$, then $|\mathrm{Out}(G)| < n^{1/2} \log_2 n$ for $n \geq 2^{14000}$.*

As mentioned earlier, the bound $n^{1/2} \log_2 n$ in Theorem 1.3 is asymptotically sharp apart from a constant factor close to 1.

The proof of Theorem 1.1 requires a careful analysis of the abelian and the nonabelian composition factors of A/G where A and G are finite groups. For this purpose for a finite group X we denote the product of the orders of the abelian and the nonabelian composition factors of a composition series for X by $a(X)$ and $b(X)$ respectively. Clearly $|X| = a(X)b(X)$.

The next result deals with $b(A/G)$ in the general case when G is transitive and in the more special situation when G is primitive.

Theorem 1.4. *Let A and G be permutation groups with $G \lhd A \leq \mathrm{S}_n$. If G is transitive, then $b(A/G) \leq n^{\log_2 n}$. If G is primitive, then $b(A/G) \leq (\log_2 n)^{2 \log_2 \log_2 n}$.*

In order to give a sharp bound for $a(A/G)$ when G is a primitive permutation group, interestingly, it is first necessary to bound $a(A)$ (for A primitive). In 1982 Pálfy [Pá] and Wolf [W] independently showed that $|A| \leq 24^{-1/3} n^{1+c_1}$ for a solvable primitive permutation group A of degree n where c_1 is the constant $\log_9(48 \cdot 24^{1/3})$ which is close to 2.24399. Equality occurs infinitely often. In fact $a(A) \leq 24^{-1/3} n^{1+c_1}$ holds [Py] for any primitive permutation group A of degree n. Using the classification theorem of finite simple groups we extend these results to the following, where for a finite group X and a prime p we denote the product of the orders of the p-solvable composition factors of X by $a_p(X)$.

Theorem 1.5. *Let $G \leq \mathrm{S}_n$ be primitive, let p be a prime divisor of n and let c_1 be as before. Then $a_p(G)|\mathrm{Out}(G)| \leq 24^{-1/3} n^{1+c_1}$.*

Wolf [W] also showed that if G is a finite nilpotent group acting faithfully and completely reducibly on a finite vector space V, then $|G| \leq |V|^{c_2}/2$ where c_2 is the constant $\log_9 32$ close to 1.57732. In order to generalize this result we set $c(X)$ to be the product of the orders of the central chief factors in a chief series of a finite group X. In particular we have $c(X) = |X|$ for a nilpotent group X. The following theorem extends Wolf's result.

Theorem 1.6. *Let $G \leq \mathrm{S}_n$ be a primitive permutation group. Then $c(G) \leq n^{c_2}/2$ where c_2 is as above.*

Some technical, module theoretic results enable us to show that if $G \lhd A \leq \mathrm{S}_n$ are transitive permutation groups, then $a(A/G) \leq 6^{n/4}$. In fact, we show that $a(A/G) \leq 4^{n/\sqrt{\log_2 n}}$ whenever $n \geq 2$. This together with Theorem 1.4 give the following.

Theorem 1.7. *We have $|A : G| \leq 4^{n/\sqrt{\log_2 n}} \cdot n^{\log_2 n}$ whenever G and A are transitive permutation groups with $G \lhd A \leq \mathrm{S}_n$ and $n \geq 2$.*

For an exponential bound in Theorem 1.7 we can have $168^{(n-1)/7}$. See [Py, Proposition 4.3] for examples of transitive p-groups (p a prime) showing that Theorem 1.7 is essentially the best one could hope for apart from the constant 4. It is also worth mentioning that a $c^{n/\sqrt{\log_2 n}}$ type bound fails in case we relax the condition $G \lhd A$ to $G \lhd\lhd A$. Indeed, if A is a Sylow 2-subgroup of S_n for n a power of 2 and G is a regular elementary abelian subgroup inside A, then $|A : G| = 2^n/2n$. The next result shows that an exponential bound in n holds in general for the index of a transitive subnormal subgroup of a permutation group of degree n.

Theorem 1.8. *Let $G \lhd\lhd A \leq \mathrm{S}_n$. If G is transitive, then $|A : G| \leq 5^{n-1}$.*

The proof of Theorem 1.8 avoids the use of the classification theorem for finite simple groups. Using the classification it is possible to replace the 5^{n-1} bound with 3^{n-1}. It would be interesting to know whether $|A : G| \leq 2^n$ holds for transitive permutation groups G and A with $G \lhd\lhd A \leq \mathrm{S}_n$.

We note that the paper contains sharp bounds for $|A : G|$, $b(A/G)$ and $a(A/G)$ in case A is a primitive permutation group of degree n and G is a transitive normal subgroup of A. These are $n^{\log_2 n}$ in the first two cases, and it is $24^{-1/3} n^{c_1}$ in the third case.

There are Galois versions of some of the above results and by [GS] these are equivalent to the corresponding group theoretic theorems.

Various corresponding results are also obtained for linear groups.

2 Methods

Various consequences of the Classification Theorem of Finite Simple Groups are used together with the Aschbacher-O'Nan-Scott Theorem. The core of the paper concerns actions of finite groups on finite vector spaces.

3 Forthcoming Research

In the spirit of Theorem 1.8 it may be possible in Theorem 1.1 to relax the condition $G \lhd A$ to $G \lhd\lhd A$ and obtain corresponding bounds for $|A : G|$. As mentioned above, it would also be interesting to know whether $|A : G| \leq 2^n$ holds for transitive permutation groups G and A with $G \lhd\lhd A \leq \mathrm{S}_n$.

References

[AG2] M. Aschbacher and R. M. Guralnick, Abelian quotients of primitive groups. *Proc. Amer. Math. Soc.* **107** (1989), 89–95.

[G2] R. M. Guralnick, Cyclic quotients of transitive groups. *J. Algebra* **234** (2000), 507–532.

[GS] R. M. Guralnick and K. Stevenson, Prescribing ramification. Arithmetic fundamental groups and noncommutative algebra (Berkeley, CA, 1999), 387–406, *Proc. Sympos. Pure Math.*, **70**, Amer. Math. Soc., Providence, RI, 2002.

[Pá] P. P. Pálfy, A polynomial bound for the orders of primitive solvable groups. *J. Algebra* **77** (1982), 127–137.

[Py] L. Pyber, Asymptotic results for permutation groups, Groups and computation, DIMACS Ser. Discrete Math. Theoret. Comput. Sci. 11 (ed. L. Finkelstein and W. M. Kantor) Amer. Math. Soc. Providence RI 1993, pp. 197–219.

[W] T. R. Wolf, Solvable and nilpotent subgroups of $GL(n, q^m)$. *Canad. J. Math.* **34** (1982), 1097–1111.

Acknowledgements

The first author was partially supported by the NSF grant DMS-1302886. The second author was supported by the MTA Rényi "Lendület" Groups and Graphs Research Group and by a Humboldt Return Fellowship. The second and third authors were supported by OTKA grants K84233 and K115799. The first and third authors would like to thank the London Mathematical Society for its support during the Durham Conference on Groups, Geometry and Combinatorics – 2001 where much of this work was done. The authors thank Miklós Abért for various eye-opening remarks.

CONJUGACY CLASSES CONTAINED IN NORMAL SUBGROUPS

A. Beltrán[1], M. J. Felipe[2] y C. Melchor[1]

[1]Departamento de Matemáticas, Universidad Jaume I, Castellón. [2]DMA-IUMPA, Universidad Politécnica de Valencia.

NOTATION: Let G be a finite group. Recall that $x^G = \{g^{-1}xg |\ g \in G\}$ is the **conjugacy class** of the element x of G, and we call its cardinal the **class size** of x. If $N \trianglelefteq G$ and $x \in N$, we say that x^G is the G**-class** of x, which is obviously contained in N.

GRAPH ASSOCIATED TO THE G-CONJUGACY CLASSES OF N, [1]

INTRODUCTION: DEFINITION OF THE GRAPH

In 1990, the graph $\Gamma(G)$ associated to the sizes of the ordinary conjugacy classes of G was introduced in [4]. We study the properties of the following subgraph of $\Gamma(G)$ regarding the G-conjugacy classes contained in N.

Definition. Let $N \trianglelefteq G$. We define the graph $\Gamma_G(N)$ as follows: the vertices are the non-central G-classes of N, and two vertices are joined by an edge if and only their sizes have a common prime divisor.

The fact that the number of connected components and the diameter of $\Gamma(G)$ are bounded does not directly imply that the corresponding for $\Gamma_G(N)$ have to be bounded too. Indeed, a prime dividing some G-class size does not need to divide $|N|$. We show that both numbers for $\Gamma_G(N)$, denoted by $n(\Gamma_G(N))$ and $d(\Gamma_G(N))$, are actually bounded.

Theorem A. *Let G be a group and let N be a normal subgroup of G. Then $n(\Gamma_G(N)) \leq 2$.*

Theorem B. *Let G be a group and let N be a normal subgroup of G.*

1. *If $n(\Gamma_G(N)) = 1$, then $d(\Gamma_G(N)) \leq 3$.*
2. *If $n(\Gamma_G(N)) = 2$, then each connected component is a complete graph.*

Theorem C. *Let G be a group and $N \trianglelefteq G$. If $n(\Gamma_G(N)) = 2$ then, either N is quasi-Frobenius with abelian kernel and complement, or $N = P \times A$ where P is a p-group and $A \leqslant \mathbf{Z}(G)$.*

We obtain the structure of N when $\Gamma_G(N)$ has no triangles. In order to prove our result we first need to study the structure of N when $\Gamma_G(N)$ has few vertices.

$\Gamma_G(N)$ WITH ONE, TWO OR THREE VERTICES, [2].

Theorem D. *If $\Gamma_G(N)$ has only one vertex, then N is a p-group for some prime p and $N/(N \cap \mathbf{Z}(G))$ is an elementary abelian p-group.*

Theorem E. *If $\Gamma_G(N)$ has two vertices and no edge, then N is a 2-group or a Frobenius group with p-elementary abelian kernel K, and complement H, which is cyclic of order q, for two different primes p and q.*

Theorem F. *If $\Gamma_G(N)$ has exactly two vertices and one edge, then one of the following possibilities holds:*

1. *N is a p-group for a prime p.*
2. *$N = P \times Q$ with $P/(\mathbf{Z}(G) \cap P)$ an elementary abelian p-group with p an odd prime, and $Q \subseteq \mathbf{Z}(G) \cap N$ and $Q \cong \mathbb{Z}_2$.*
3. *N is a Frobenius group with p-elementary abelian kernel K and complement $H \cong \mathbb{Z}_q$ for some distinct primes p and q.*

Theorem G. *If $\Gamma_G(N)$ has three vertices in a line, then $\mathbf{Z}(G) \cap N = 1$ and one of the following cases is satisfied:*

1. *N is a 2-group of exponent at most 4.*
2. *$N = P \times Q$, where P and Q are elementary abelian p and q-groups.*
3. *N is a Frobenius group with complement isomorphic to $\mathbb{Z}_q$, $\mathbb{Z}_{q^2}$ or Q_8. In the former case, the kernel of N is a p-group with exponent $\leq p^2$ and in the two latter cases, the kernel of N is p-elementary abelian.*

Theorem H. *If $\Gamma_G(N)$ has three vertices and one edge, then N is a $\{p, q\}$-group for two primes p and q. Furthermore, either N is a p-group, or N is a quasi-Frobenius group with abelian kernel and complement. In this case, $|N \cap \mathbf{Z}(G)| = 1$ or 2.*

$\Gamma_G(N)$ IS EXACTLY A TRIANGLE, [2].

Theorem I. *If $\Gamma_G(N)$ consists of exactly one triangle, then one of the following holds:*

1. *N is a p-group for some prime p.*
2. *$N = P \times Q$, with either P p-elementary abelian and Q q-elementary abelian for some primes p and q, and $\mathbf{Z}(G) \cap N = 1$ or P a p-group for a prime $p \neq 3$, and $Q \subseteq \mathbf{Z}(G) \cap N$, $Q \cong \mathbb{Z}_3$ and $P/(\mathbf{Z}(G) \cap P)$ has exponent p.*
3. *$N = PQ$, where P is a Sylow p-subgroup, $p \neq 2$ and Q is a Sylow 2-subgroup of N. In addition, P has exponent p, $|\mathbf{Z}(G) \cap N| = 2$ and $Q/(\mathbf{Z}(G) \cap N)$ is 2-elementary abelian.*
4. *Either N is a Frobenius group with complement $\mathbb{Z}_q$, $\mathbb{Z}_{q^2}$ or Q_8 for a prime q, or there are two primes p and q such that $N/\mathbf{O}_p(N)$ is a Frobenius group of order pq and $\mathbf{O}_p(N)$ has exponent p. In this case, $\mathbf{Z}(G) \cap N = 1$.*
5. *$N \cong A_5$ and $G = (N \times K)\langle x \rangle$ for some $K \leq G$ and $x \in G$, with $x^2 \in N \times K$ and $G/K \cong N\langle x \rangle \cong S_5$.*

$\Gamma_G(N)$ WITHOUT TRIANGLES, [2].

Theorem J. *If $\Gamma_G(N)$ has no triangles, then N is a $\{p, q\}$-group and satisfies one of these properties:*

1. *N is a p-group.*
2. *$N = P \times Q$ with P a p-group and $Q \subseteq \mathbf{Z}(G) \cap N$, $Q \cong \mathbb{Z}_2$.*
3. *$N = P \times Q$ with P a p-group and Q a q-group both elementary abelian with p and q odd primes. In this case $\mathbf{Z}(G) \cap N = 1$.*
4. *N is a quasi-Frobenius group with abelian kernel and complement and $\mathbf{Z}(G) \cap N \cong \mathbb{Z}_2$.*
5. *N is a Frobenius group with complement isomorphic to $\mathbb{Z}_q$, $\mathbb{Z}_{q^2}$ or Q_8. In the first case, the kernel of N is a p-group with exponent less or equal than p^2 and in the two latter cases, the kernel of N is p-elementary abelian.*

LANDAU'S THEOREM ON CONJUGACY CLASSES FOR NORMAL SUBGROUPS, [3].

Theorem K. *Let $s, n \in \mathbb{N}$ such that $s, n \geq 1$. There exists at most a finite number of isomorphism classes of finite groups G which contains a normal subgroup N such that $|G : N| = n$ and N has exactly s non-central G-classes. Moreover, if G and N satisfy such condition, then*

$$|G| < n^{2^s+1}(s+1)\prod_{i=0}^{s-1}(s+1-i)^{2^{s-1-i}} \quad \text{and} \quad |N| < n^{2^s}(s+1)\prod_{i=0}^{s-1}(s+1-i)^{2^{s-1-i}}$$

Theorem L. *Let $N \trianglelefteq G$ with $|G : N| = n$. Suppose that G has exactly only one non-central G-conjugacy class. Then $|G| < n(n + 1)^2$.*

Number of groups with a normal subgroup having exactly one non-central G-class (by using [5]) and a comparison of the bounds obtained in Theorems K and L.

$\lvert G:N\rvert$	$\lvert G\rvert \leq 4n^3$	$\lvert G\rvert \leq n(n+1)^2$	Number of groups
2	32	18	3
3	108	48	2
4	256	100	21
5	500	180	0
6	864	294	16
7	1372	448	1

REFERENCES

[1] A. Beltrán, M.J. Felipe and C. Melchor, Graphs associated to conjugacy classes of normal subgroups in finite groups. J. Algebra, **443** (2015), 335-348.

[2] A. Beltrán, M.J. Felipe and C. Melchor, Triangles in the graph of conjugacy classes of normal subgroups. Monatshetfte für Mathematik, 10.1007/s00605-015-0866-9.

[3] A. Beltrán, M.J. Felipe and C. Melchor, Landau's Theorem on conjugacy classes for normal subgroups. Submitted (2016).

[4] E.A. Bertram, M. Herzog, A. Mann, On a graph related to conjugacy classes of groups. Bull. London Math. Soc. **22** (6) (1990), 569-575.

[5] The GAP Group, GAP - Groups, Algorithms and Programming, Vers. 4.7.7; 2015. (http://www.gap-system.org)

This research is partially supported by a predoctoral grant of Universitat Jaume I. The research of the first and second authors is supported by the Valencian Government, Proyecto PROMETEOII/2015/011. The first and the third authors are also partially supported by grant P11B2015-77.

Square-free class sizes in mutually permutable products

M. J. Felipe · A. Martínez-Pastor · V. M. Ortiz-Sotomayor[1]

Abstract

Over the last years, many authors have investigated the influence of conjugacy class sizes on the structure of finite groups. At the same time, the study of groups factorised as product of subgroups has been object of increasing interest, specially when they are connected by certain permutability properties.

The purpose of this poster is to present new achievements which combine both current perspectives in Finite Group Theory. A first approach to this topic can be found either in [1] or in [6], although the literature in this framework is sparse.

Basic concepts and terminology

In the sequel, all groups considered are finite. We deal with factorised groups whose factors are connected by certain permutability properties (see [2]). Two subgroups A and B of a group G are called **mutually permutable** if A permutes with every subgroup of B and B permutes with every subgroup of A.

The notation here is as follows: the set $x^G := \{g^{-1}xg \,:\, g \in G\}$ is called **conjugacy class** of the element $x \in G$. We denote by $|x^G|$ the **size of the conjugacy class** x^G. If p is a prime, we say that $x \in G$ is a ***p*-regular element** if its order is not divisible by p. The remainder notation is standard in the framework of group theory.

Introduction

The earlier starting point of our investigation can be traced in the paper of Chillag and Herzog in 1990 ([3]), where several results were proved about the global structure of a group if some arithmetical information is known about its conjugacy class sizes. In particular, they handled the situation when all elements of the group have square-free conjugacy class sizes, using the classification theorem of finite simple groups (CFSG).

In [4], Cossey and Wang considered conjugacy class sizes not divisible by p^2, for certain fixed prime p. Later on, this study was improved by Liu, Wang, and Wei in [6], by replacing conditions for all conjugacy classes by those referring only to conjugacy classes of either p-regular elements or prime power order elements.

These authors also first analysed some preliminary results in factorised groups which were extended, in 2012, by Ballester-Bolinches, Cossey and Li ([1]), through products of mutually permutable subgroups.

Main results

In 2014, Qian and Wang ([7]) have gone a step further in the above study by considering just conjugacy class sizes of p-regular elements of prime power order (although not in factorised groups), as the following theorem shows.

Theorem 1. For a fixed prime p with $(p-1, |G|) = 1$, if p^2 does not divide $|x^G|$ for any p-regular element $x \in G$ of prime power order, then G is solvable, p-nilpotent and the Sylow p-subgroups of $G/\operatorname{O}_p(G)$ are elementary abelian.

Motivated by the previous development, our first result generalises this theorem through products of two mutually permutable subgroups. We point out that both results apply the CFSG.

Theorem A ([5]). Let $G = AB$ be the product of the mutually permutable subgroups A and B and let p be a prime with $(p-1, |G|) = 1$. If p^2 does not divide $|x^G|$, for any p-regular element $x \in A \cup B$ of prime power order, then G is solvable, p-nilpotent and the Sylow p-subgroups of $G/\operatorname{O}_p(G)$ are elementary abelian.

On the other hand, Ballester-Bolinches, Cossey and Li proved in [1] the next result.

Theorem 2. Let $G = AB$ be the product of the mutually permutable subgroups A and B and let p be a prime. Suppose that for every p-regular element $x \in A \cup B$, $|x^G|$ is not divisible by p^2. Then the order of the Sylow p-subgroups of every chief factor of G is at most p. In particular, if G is p-solvable, we have that G is p-supersolvable.

The second assertion of the above theorem can be generalised for p-regular elements of prime power order as follows.

Theorem B ([5]). Let $G = AB$ be the product of the mutually permutable subgroups A and B and let p be a prime. Suppose that for every p-regular element $x \in A \cup B$ of prime power order, $|x^G|$ is not divisible by p^2. Then if G is p-solvable, we have that G is p-supersolvable.

Regarding the first assertion in Theorem 2, at least we know that it remains true when considering p-regular elements of prime power order if p is the largest prime dividing $|G|$, although the general case is still an open question.

If the assumptions of Theorem B hold for every prime, we get the supersolvability of G and some information about the structure of the Sylow subgroups of $G/\operatorname{F}(G)$.

Theorem C ([5]). Let $G = AB$ be the product of the mutually permutable subgroups A and B. Suppose that for every prime p and for every p-regular element $x \in A \cup B$ of prime power order, $|x^G|$ is not divisible by p^2. Then G is supersolvable and $G/\operatorname{F}(G)$ has elementary abelian Sylow subgroups.

If we impose to the previous result the stronger condition that each prime power order element of the factors has square-free conjugacy class size, then we obtain some additional information about the derived subgroup of G.

Theorem D ([5]). Let $G = AB$ be the product of the mutually permutable subgroups A and B. If every prime power order element $x \in A \cup B$ has square-free conjugacy class size, then G is supersolvable, $G/\operatorname{F}(G)$ has elementary abelian Sylow subgroups and G' is abelian.

Finally, imposing the hypotheses of Theorem C to all p-regular elements of the factors (not only to those of prime power order), we can bound the orders of the Sylow subgroups of $G/\operatorname{F}(G)$.

Theorem E ([5]). Let $G = AB$ be the product of the mutually permutable subgroups A and B. If for every prime p and for every p-regular element $x \in A \cup B$, $|x^G|$ is not divisible by p^2, then G is supersolvable and $G/\operatorname{F}(G)$ has elementary abelian Sylow p-subgroups of order at most p^2, for each prime divisor p of $|G|$.

It is not difficult to find examples which show that the stronger conditions of the previous two results are necessary in contrast to those in Theorem C. We include in [5] some examples that illustrate the scope of the results presented in this poster.

Acknowledgements. The first author has been supported by Proyecto Prometeo II/2015/011, and the third author has been partially supported by Proyecto Prometeo II/2013/013, Generalitat Valenciana, Spain. The results in this poster are part of the third author's Ph. D. thesis at Universidad Politécnica de Valencia (Spain).

References

[1] Ballester-Bolinches, A., Cossey, J. and Li, Y., Mutually permutable products and conjugacy classes, *Monatsh Math*, 170: 305-310, 2012.

[2] Ballester-Bolinches, A., Esteban-Romero, R. and Asaad, M., *Products of finite groups*, vol. 53 of de Gruyter Expositions in Mathematics, Berlin, 2010.

[3] Chillag, D. and Herzog, M., On the length of the conjugacy classes of finite groups, *J. Algebra*, 131: 110-125, 1990.

[4] Cossey, J. and Wang, Y., Remarks on the length of conjugacy classes of finite groups, *Communications in Algebra*, 27: 4347-4353, 1999.

[5] Felipe, M.J., Martínez-Pastor, A. and Ortiz-Sotomayor, V.M., Square-free class sizes in mutually permutable products, *preprint*, 2016.

[6] Liu, X., Wang, Y. and Wei, H., Notes on the length of conjugacy classes of finite groups, *J. Pure and Applied Algebra*, 196: 111-117, 2005.

[7] Qian, G. and Wang, Y., On conjugacy class sizes and character degrees of finite groups, *Journal of Algebra and Its Applications*, 13: 1350100 (9 pages), 2014.

[1]**Contact Information:**
Instituto Universitario de Matemática Pura y Aplicada (IUMPA),
Universidad Politécnica de Valencia, Camino de Vera s/n, 46022,
Valencia, Spain.

Emails: mfelipe@mat.upv.es, anamarti@mat.upv.es, vicorso@doctor.upv.es

q-Tensor Squares of f. g. Nilpotent Groups, $q \geq 0$

Noraí R. Rocco
Joint work with Eunice Rodrigues
norai@unb.br

ISCHIA GROUP THEORY 2016 – MARCH, 29TH - APRIL, 2ND

Abstract

We extend to the $q-$tensor square $G \otimes^q G$ of a group G, q a non- negative integer, some structural results found in [2] concerning the non-abelian tensor square $G \otimes G$ ($q = 0$). The results are applied to the computation of $G \otimes^q G$ for G a finitely generated nilpotent group. We also generalise to all $q \geq 0$ results from [1] regarding the minimal number of generators of the non-abelian tensor square $G \otimes G$ when G is a $n-$generator nilpotent group of class 2. Finally, we determine the $q-$tensor square of $\mathcal{N}_{n,2}$, the free $n-$generator nilpotent group of class 2, for all $q \geq 0$.

Introduction

Let G and G^φ be groups, isomorphic via $\varphi : g \mapsto g^\varphi$, for all $g \in G$, and let $\nu(G)$ be the group

$$\nu(G) := \left\langle G \cup G^\varphi \mid [g, h^\varphi]^k = [g^k, (h^k)^\varphi] = [g, h^\varphi]^{k^\varphi}, \forall g, h, k \in G \right\rangle .$$

Then the subgroup $\Upsilon(G) = [G, G^\varphi] \leq \nu(G)$ is isomorphic to the non-abelian tensor square $G \otimes G$; we write $[g, h^\varphi]$ for $g \otimes h$, $\forall g, h \in G$.

Now, if $q \geq 1$ then let $\widehat{\mathcal{G}} = \{\widehat{k} \mid k \in G\}$ be a set of symbols, one for each element of G (for $q = 0$ set $\widehat{\mathcal{G}} = \emptyset$). Let $F(\widehat{\mathcal{G}})$ be the free group over $\widehat{\mathcal{G}}$. As G, G^φ are embedded into $\nu(G)$, we identify their elements by their respective images in the free product $\nu(G) * F(\widehat{\mathcal{G}})$. Let J denote the normal closure in $\nu(G) * F(\widehat{\mathcal{G}})$ of the following elements, for all $\widehat{k}, \widehat{k_1} \in \widehat{\mathcal{G}}$ and $g, h \in G$:

$$g^{-1}\,\widehat{k}\,g\;(\widehat{k^g})^{-1}; \tag{1}$$

$$(h^\varphi)^{-1}\,\widehat{k}\,h^\varphi\;(\widehat{k^h})^{-1}; \tag{2}$$

$$(\widehat{k})^{-1}[g, h^\varphi]\,\widehat{k}\;[g^{k^q}, (h^{k^q})^\varphi]^{-1}; \tag{3}$$

$$(\widehat{k})^{-1}\,\widehat{kk_1}\;(\widehat{k_1})^{-1}\Big(\prod_{i=1}^{q-1}[k, (k_1^{-i})^\varphi]^{k^{q-1-i}}\Big)^{-1}; \tag{4}$$

$$[\widehat{k}, \widehat{k_1}]\;[k^q, (k_1^q)^\varphi]^{-1}; \tag{5}$$

$$\widehat{[g, h]}\;[g, h^\varphi]^{-q}. \tag{6}$$

Define $\nu^q(G) := (\nu(G) * F(\widehat{\mathcal{G}}))/J$

For $q = 0$ the set of all relations (1) to (6) is empty; hence, $\nu^0(G) \cong \nu(G)$.

Some Relevant Sections of $\nu^q(G)$

1. G and G^φ are embedded into $\nu^q(G)$, for all $q \geq 0$
2. Set $T := [G, G^\varphi]$ and $\mathfrak{G} := \langle \widehat{\mathcal{G}} \rangle$.
3. $\Upsilon^q(G) := T\mathfrak{G} \trianglelefteq \nu^q(G)$ and $\nu^q(G) = G^\varphi \cdot (G \cdot \Upsilon^q(G))$
4. $\Upsilon^q(G) \cong G \otimes^q G$, for all $q \geq 0$ ([6] and [5])
   (This sets out a *"hat-commutator"* approach to the q-tensor square).
5. $G \wedge^q G \cong \Upsilon^q(G)/\Delta^q(G)$, where $\Delta^q(G) = \langle [g, g^\varphi] \mid g \in G \rangle$.
6. Let $\rho' : \Upsilon^q(G) \to G$ be induced by $[g, h^\varphi] \mapsto [g, h]$, $\widehat{k} \mapsto k^q$.
   $\Rightarrow$ $\mathrm{Ker}(\rho')/\Delta^q(G) \cong H_2(G, \mathbb{Z}_q)$.

Structural Results and Computations

[5, Thm 3.1] Let $d = \gcd(q, n)$. Then

$$C_\infty \otimes^q C_\infty \cong C_\infty \times C_q,$$

$$C_n \otimes^q C_n \cong \begin{cases} C_n \times C_d, & \text{if } \ d \text{ is odd,} \\ C_n \times C_d, & \text{if } \ d \text{ is even and either } 4|n \text{ or } 4|q; \\ C_{2n} \times C_{d/2}, & \text{otherwise.} \end{cases}$$

[[5, Cor. 2.16] Let $G = N \times H$, $\overline{N} = N/N'N^q$ and $\overline{H} = H/H'H^q$.

1. $\nu^q(G) = \langle N, N^\varphi, \widehat{\mathcal{N}} \rangle \times [N, H^\varphi][H, N^\varphi] \times \langle H, H^\varphi, \widehat{\mathcal{H}} \rangle$;
2. $\langle H, H^\varphi, \widehat{\mathcal{H}} \rangle \cong \nu^q(H)$; $\qquad \langle N, N^\varphi, \widehat{\mathcal{N}} \rangle \cong \nu^q(N)$.
3. $\Upsilon^q(G) = \Upsilon^q(N) \times [N, H^\varphi][H, N^\varphi] \times \Upsilon^q(H)$;
4. $[N, H^\varphi] \cong \overline{N} \otimes_{\mathbb{Z}_q} \overline{H} \cong [H, N^\varphi]$.

[7, Cor. 2.2] Let G be a group and $g, h \in G$. Then

1. $[G', G^\varphi] = [G, G'^\varphi]$;
2. $[G', Z(G)^\varphi] = 1$;
3. If $gG' = hG'$ then $[g, g^\varphi] = [h, h^\varphi]$;
4. If $o'(x)$ denotes the order of a coset $xG' \in G/G'$, then $[g, h^\varphi][h, g^\varphi]$ has order dividing the $\gcd(q, o'(g), o'(h))$;
5. The order of $[h, h^\varphi]$ divides the $\gcd(q, o'(h)^2, 2o'(h))$.

Next theorem generalises, to all $q \geq 0$, Proposition 2.2 in [2].

[7, Thm 2.8] Let G be a group and assume that G^{ab} is f.g.

1. If $q \geq 1$ and q is odd, then $\Upsilon^q(G) \cong \Delta^q(G^{ab}) \times (G \wedge^q G)$;
2. For $q = 0$ or $q \geq 2$ and q even, if $r_2(G^{ab}) = 0$ or if G' has a complement in G, then also $\Upsilon^q(G) \cong \Delta^q(G^{ab}) \times (G \wedge^q G)$;
3. For $q \geq 2$ and q even, if $r_2(G^{ab}) = 0$, then $\Delta^q(G)$ is a homocyclic abelian group of exponent q, of rank $\binom{t+1}{2}$;
4. If G^{ab} is free abelian of rank t, then the conclusion of the previous item holds for all $q \geq 1$, while $\Delta^q(G)$ is free abelian of rank $\binom{t+1}{2}$ if $q = 0$.

[7, Cor. 2.10, 2.11] Let F_n be the free group of rank n and $\mathcal{N}_{n,c} = F_n/\gamma_{c+1}(F_n)$. Then, for $q \geq 1$, we have

(i) $F_n \otimes^q F_n \cong C_q^{\binom{n+1}{2}} \times (F_n)'(F_n)^q$;

(ii) $\mathcal{N}_{n,c} \otimes^q \mathcal{N}_{n,c} \cong C_q^{\binom{n+1}{2}} \times \frac{(F_n)'(F_n)^q}{\gamma_{c+1}(F_n)^q\gamma_{c+2}(F_n)}$;

If $q = 0$ then

(iii) ([4, Proposition 6]) $F_n \otimes F_n \cong C_\infty^{\binom{n+1}{2}} \times (F_n)'$.

(iv) ([3, Corollary 1.7]) $\mathcal{N}_{n,c} \otimes \mathcal{N}_{n,c} \cong C_\infty^{\binom{n+1}{2}} \times (\mathcal{N}_{n,c+1})'$.

[7, Thm 3.2] Let G be a nilpotent group of class 2 with $d(G) = n$.

(i) ([1, Theorem 3.1]) $d([G, G^\varphi]) \leq \frac{n(n^2+3n-1)}{3}$;

(ii) $d(G \otimes^q G) \leq \frac{n(n^2+3n+2)}{3}$, for all $q \geq 0$;

(iii) If G has finite exponent and $\gcd(q, \exp(G)) = 1$, then $d(G \otimes^q G) \leq n^2$.

[7, Prop. 3.3] Let $\mathcal{N}_{n,2} = F_n/\gamma_3(F_n)$ be the free nilpotent group of rank $n > 1$ and class 2. Then,

(i) ([1, Theorem 3.2]) $\mathcal{N}_{n,2} \otimes \mathcal{N}_{n,2}$ is free abelian of rank $\frac{1}{3}n(n^2 + 3n - 1)$. More precisely,

$$\mathcal{N}_{n,2} \otimes \mathcal{N}_{n,2} \cong \Delta(F_n^{ab}) \times H_2(\mathcal{N}_{n,2}, \mathbb{Z}) \times \mathcal{N}'_{n,2}.$$

(ii)

$$\mathcal{N}_{n,2} \otimes^q \mathcal{N}_{n,2} \cong (C_q)^{(\binom{n+1}{2}+M_n(3))} \times \mathcal{N}'_{n,2}\mathcal{N}^q_{n,2},$$

where $M_n(3) = \frac{1}{3}(n^3 - n)$ is the $q-$rank of $\gamma_3(\mathcal{N}_{n,2})/\gamma_3(\mathcal{N}_{n,2})^q\gamma_4(\mathcal{N}_{n,2})$, according to the Witt's formula. Consequently, for $q > 1$,

$$d(\mathcal{N}_{n,2} \otimes^q \mathcal{N}_{n,2}) = \frac{1}{3}(n^3 + 3n^2 + 2n).$$

REFERENCES

[1] Bacon, M., *On the non-abelian Tensor Square of a Nilpotent Group of Class Two*, Glasgow Math. J., **3** (1994), 291–295.

[2] Blyth, R. D., Fumagalli, F., Morigi, M., *Some structural results on the non- abelian tensor square of groups*, J. Group Theory, **13**, No.1 (2010), 83–94.

[3] Blyth, R. D., Moravec, P., Morse, R. F., *On the nonabelian tensor squares of free nilpotent groups of finite rank*. In *Computational Group Theory and the Theory of Groups*, Contemporary Mathematics 470 (American Mathematical Society, 2008), pp. 27–44.

[4] Brown, R., Johnson, D. L., Robertson, E. F., *Some computations of non-abelian tensor products of groups*, J. Algebra bf 111 (1987), 177–202.

[5] Bueno, T. P. and Rocco, N. R., *On the q-tensor square of a group*, J. Group Theory, **14** (2011), 785–805.

[6] Ellis, G., *Tensor Product and q-crossed Modules*, J. London Math. Soc., **2** **(51)** No.2 (1995), 243–258.

[7] Rocco, N. R. and Rodrigues, E. C. P., *The q-tensor square of finitely generated nilpotent groups*, $q \geq 0$, preprint (2016), 12 pp. Available at http://arxiv.org/abs/1603. 05424

NEIGHBORHOOD RADIUS ESTIMATION FOR ARNOLDS MINIVERSAL DEFORMATIONS OF COMPLEX AND p-ADIC MATRICES

Mohammed A. Salim
Department of Mathematical Sciences
United Arab Emirates University, U.A.E.

جامعة الإمارات العربية المتحدة
United Arab Emirates University

Abstract

V.I. Arnold [1] constructed a *miniversal deformation* of a given square complex matrix A, i.e., a simple normal form to which all complex matrices B in a neighborhood U of A can be reduced by similarity transformations that smoothly depend on the entries of B.
D.M. Galin [2] constructed a miniversal deformation of square real matrices.

- We calculate the radius of the neighborhood U, which is important for applications.
- **A.A. Mailybaev [3]** constructed a reducing similarity transformation in the form of Taylor series; we construct this transformation by another method.
- We extend Arnold's normal form to matrices over the field $\mathbb{Q}_p$ of p-adic numbers and the field $\mathbb{K}((T))$ of Laurent series over a field $\mathbb{K}$.

Motivation

The reduction of a complex matrix A to its Jordan form is an unstable operation: both the Jordan form and a reduction transformation depend discontinuously on the entries of the original matrix. **V.I. Arnold [1]** supposed without restriction that A is a Jordan canonical matrix and have reduced all matrices B in a neighborhood U of A to the form B_{arn} by a smooth similarity transformation that acts identically on A. Many applications of miniversal deformations are based on the fact that the spectrum of $B \in U$ and B_{arn} coincide but B_{arn} has a simple form.

Example: Arnold's miniversal deformation of $J_5(\lambda) \oplus J_3(\lambda) \oplus J_3(\mu)$

Given the Jordan matrix $J := J_5(\lambda) \oplus J_3(\lambda) \oplus J_3(\mu)$ $(\lambda \neq \mu)$. All complex matrices $J + X$ that are sufficiently close to J can be simultaneously reduced by some transformation

$$J + X \mapsto \mathcal{S}(X)^{-1}(J + X)\mathcal{S}(X), \qquad \begin{array}{c}\mathcal{S}(X) \text{ is nonsingular and} \\ \text{analytic at zero, } \mathcal{S}(\mathbf{0}) = I_n\end{array}$$

to the form

$$\left[\begin{array}{ccccc|ccc|ccc}
\lambda & 1 & & & & & & & & & \\
 & \lambda & 1 & & & & & & & & \\
 & & \lambda & 1 & & & & & & \mathbf{0} & \\
 & & & \lambda & 1 & & & & & & \\
* & * & * & * & \lambda+* & * & * & * & & & \\
\hline
* & & & & & \lambda & 1 & & & & \\
* & & & & & & \lambda & 1 & & \mathbf{0} & \\
* & & & & & * & * & \lambda+* & & & \\
\hline
 & & & & & & & & \mu & 1 & \\
 & & \mathbf{0} & & & & \mathbf{0} & & & \mu & 1 \\
 & & & & & & & & * & * & \mu+*
\end{array}\right]$$

in which the stars of $\mathcal{D}$ represent elements that depend analytically on the entries of X.

Frobenius canonical matrix

- A Frobenius canonical block is a matrix

$$\Phi_m(p) := \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \\ -c_m & \dots & -c_2 & -c_1 \end{bmatrix} \qquad (m\text{-by-}m)$$

whose characteristic polynomial $x^m + c_1x^{m-1} + \dots + c_m \in \mathbb{F}[x]$ is an integer power of a polynomial $p(x)$ that is irreducible over $\mathbb{F}$.

- A Frobenius canonical matrix for similarity is a direct sum of Frobenius blocks:

$$\Phi := \bigoplus_{i=1}^{t} \left(\Phi_{m_{i1}}(p_i) \oplus \Phi_{m_{i2}}(p_i) \oplus \dots \oplus \Phi_{m_{ik_i}}(p_i) \right) \tag{1}$$

we suppose that $m_{i1} \geq \dots \geq m_{ik_i}$. (Each square matrix over an arbitrary field is similar to a matrix of the form Φ, which is uniquely determined, up to permutation of direct summands.)

Theorem 1

Let $\mathbb{F}$ be one of the fields: $\mathbb{C}$, $\mathbb{R}$, the field $\mathbb{Q}_p$ of p-adic numbers, or the field $\mathbb{K}((T))$ of Laurent series over a field $\mathbb{K}$. Let Φ be the $n \times n$ Frobenius canonical matrix (1) over $\mathbb{F}$. *Then there exists a neighborhood $U \subset \mathbb{F}^{n\times n}$ of $\mathbf{0}_n$ such that all matrices $\Phi + X$ with $X \in U$ can be simultaneously reduced by some transformation*

$$\Phi + X \mapsto \mathcal{S}(X)^{-1}(J + X)\mathcal{S}(X), \qquad \begin{array}{c}\mathcal{S}(X) \text{ is nonsingular and} \\ \text{continuous, } \mathcal{S}(\mathbf{0}) = I_n\end{array} \tag{2}$$

to the form $\Phi + \mathcal{D}$, in which

$$\mathcal{D} := \bigoplus_{i=1}^{t} \begin{bmatrix} \mathbf{0}^{\downarrow}_{m_{i1}} & \mathbf{0}^{\downarrow} & \dots & \mathbf{0}^{\downarrow} \\ \mathbf{0}^{\leftarrow} & \mathbf{0}^{\downarrow}_{m_{i2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0}^{\downarrow} \\ \mathbf{0}^{\leftarrow} & \dots & \mathbf{0}^{\leftarrow} & \mathbf{0}^{\downarrow}_{m_{ik_i}} \end{bmatrix}, \tag{3}$$

$$\mathbf{0}^{\downarrow} := \begin{bmatrix} & 0 & \\ * & \cdots & * \end{bmatrix}, \qquad \mathbf{0}^{\leftarrow} := \begin{bmatrix} * & \\ \vdots & 0 \\ * & \end{bmatrix},$$

and $\mathbf{0}^{\downarrow}_m$ denotes the matrix $\mathbf{0}^{\downarrow}$ of size $m \times m$. The stars of $\mathcal{D}$ represent elements that depend continuously on the entries of X.

Theorem 2

- Denote by $\mathcal{D}(\mathbb{F})$ the vector space of all matrices obtained from $\mathcal{D}$ in (3) by replacing its stars with elements of $\mathbb{F}$.
- For each $n \times n$ matrix unit E_{ij}, we fix an $n \times n$ matrix F_{ij} such that

$$E_{ij} + F_{ij}^T A - AF_{ij} \in \mathcal{D}(\mathbb{F}).$$

Then the neighborhood U can be taken as follows:

$$U := \left\{ X \in \mathbb{F}^{n\times n} \;\middle|\; \|X\| < \frac{1}{48\sqrt{n}(a+1)f^2} \right\} \tag{4}$$

in which

$$a := \|A\|, \qquad f := \max\left\{ \sum_{i,j} \|F_{ij}\|, \tfrac{1}{3} \right\},$$

$$\|M\| := \sqrt{\sum |m_{ij}|^2} \qquad \text{for all } M = [m_{ij}] \in \mathbb{F}^{n\times n}.$$

Theorem 3

For each $X \in U$ from (4), construct a sequence

$$M_1 := X, \; M_2, \; M_3, \dots$$

of $n \times n$ matrices as follows: if $M_k = [m_{ij}^{(k)}]$ has been constructed, then

$$M_{k+1} := -A + (I_n - C_k)^{-1}(A + M_k)(I_n - C_k),$$

where

$$C_k := \sum_{i,j} m_{ij}^{(k)} F_{ij}.$$

Then the matrix function $\mathcal{S}(X)$ in (2) can be taken as the infinite product

$$\mathcal{S}(X) := (I_n - C_1)(I_n - C_2)(I_n - C_3)\cdots$$

and for each $X \in U$

$$\|\mathcal{S}(X) - I_n\| < -1 + (1 + 1/2)(1 + 1/4)(1 + 1/8)\cdots$$
$$\|D(X)\| \leq 1/(4f).$$

References

[1] **V.I. Arnold**, On matrices depending on parameters, *Russian Math. Surveys* 26 (2) (1971) 29–43.
[2] **D.M. Galin**, On real matrices depending on parameters, *Uspehi Mat. Nauk* 27 (1) (1972) 241–242 (in Russian).
[3] **A.A. Mailybaev**, Transformation of families of matrices to normal forms and its application to stability theory, *SIAM J. Matrix Anal. Appl.* 21 (1999) 396–417.

This is a joint work with Victor A. Bovdi and Vladimir V. Sergeichuk

Groups with all subgroups of infinite rank (S-)semipermutable

Roberto Ialenti
Dipartimento di Matematica e Applicazioni
Università Federico II di Napoli
roberto.ialenti@unina.it

Francesca Spagnuolo *
Departament d'Àlgebra
Universitat de València
francesca.spagnuolo@uv.es

* *This is joint work with Martyn R. Dixon, University of Alabama, and Sergio Camp Mora, Universitat Politècnica de València*

Ischia Group Theory 2016

Introduction

Definition. Let G be a group. A subgroup H of G is called *semipermutable* in G if HK is a subgroup of G, for every subgroup K of G such that $\pi(H) \cap \pi(K) = \emptyset$.

Definition. Let G be a group. A subgroup H of G is called *S-semipermutable* in G if for every prime number $p \notin \pi(H)$, HP is a subgroup of G for every Sylow p-subgroup P of G.

Proposition. *Let G be a group and let H and K be subgroups of G such that $H \leq K$.*

- *If H is semipermutable in G, then H is semipermutable in K.*
- *If H is S-semipermutable in G, then H is S-semipermutable in K.*

The property of being (S)-semipermutable is not closed by subgroups and homomorphic images, as we can see in the following examples.

1. In S_4 any subgroup of order 6 is semipermutable. Let X be one of them, then $X \cap A_4$ is not semipermutable in A_4.
2. Let $G = A_4 \times C_3$, let x be an element of order 2 in A_4 and let $X = \langle x \rangle \times C_3$. H is semipermutable in G but X/C_3 is not semipermutable G/C_3.

In a locally finite group G, semipermutability and S-semipermutability can be controlled by the behaviour of the cyclic subgroups of G, as the following lemmas show.

Lemma 1. *Let G be a locally finite group. Every subgroup of G is semipermutable in G if and only if for every prime numbers p and q, with $p \neq q$, and for every p-element x and q-element y of G, $\langle x \rangle \langle y \rangle$ is a subgroup of G.*

Lemma 2. *Let G be a locally finite group. Every subgroup of G is S-semipermutable in G if and only if for every prime numbers p and q, with $p \neq q$, and for every p-element x of G and Sylow q-subgroup Q of G, $\langle x \rangle Q$ is a subgroup of G.*

Semipermutable case

The following two lemmas allow us to prove the main theorem on semipermutable case.

Lemma 3. *Let G be a locally finite group with infinite rank whose subgroups of infinite rank are semipermutable. If G has a r-subgroup S with infinite rank and a p-element x, where p and r are different prime numbers, then for every subgroup H of S, $H\langle x \rangle$ is a subgroup of G. In particular, $S\langle x \rangle$ is a $\{p, r\}-$group.*

Lemma 4. *Let G be a locally finite group with infinite rank whose subgroups of infinite rank are semipermutable. Let p, q, r be pairwise different prime numers and let S be a r-subgroup of G with infinite rank. Then, if x is a p-element and y is a q-element of G, $\langle x \rangle \langle y \rangle$ is a subgroup of G.*

The following lemma allows us to restrict our attention to countable groups with infinite rank.

Lemma 5. *Let G be a locally finite group with infinite rank. If every countable subgroup of G with infinite rank has all subgroups semipermutable, then all subgroups of G are semipermutable.*

Theorem A. *Let G be a locally finite group with infinite rank whose subgroups of infinite rank are semipermutable. Then every subgroup of G is semipermutable.*

Sketch of the proof. Let $x, y \in G$ with $o(x) = p^\alpha$ and $o(y) = q^\beta$, with $p \neq q$. Let prove that $\langle x \rangle \langle y \rangle$ is a subgroup of G.

If G has min-p for every p, we can construct the following series of normal subgroups of G

$$A_1 \gneq A_2 \gneq \ldots \gneq A_n \gneq \ldots$$

such that $\bigcap_{n\geq 1} A_n = \{1\}$ and the rank of A_i is infinite. Furthermore $\langle x \rangle A_i$ is semipermutable for every $i \geq 1$.
There exists a positive integer j such that for every $i \geq j$, $q \notin \pi(A_i)$. So for every $i \geq j$, $(\langle x \rangle A_i)\langle y \rangle$ is a subgroup of G. We prove that

$$\langle x \rangle \langle y \rangle = \bigcap_{i\geq 1} \langle x \rangle \langle y \rangle A_i.$$

So $\langle x \rangle \langle y \rangle$ is a subgroup of G.

Then let suppose that there exists a Sylow r-subgroup S of G with infinite rank, for some prime number r.

If $r \neq p, q$, then by Lemma 4 $\langle x \rangle \langle y \rangle$ is a subgroup of G.

Let suppose that $r = p$. We proved that every p-element of G is contained in a Sylow p-subgroup with infinite rank. Then the statement follows from Lemma 3. □

S-semipermutable case

Theorem B. *Let G be a locally finite group with infinite rank whose subgroups of infinite rank are S-semipermutable. If G has min-p for every p, then every subgroup of G is S-semipermutable*

Proof. Let x be a p-element of G and let Q be a Sylow q-subgroup of G, where p and q are different prime numbers. We want to prove that $\langle x \rangle Q$ is a subgroup of G.
By Theorem 3.5.15 of [?], G has a locally soluble normal subgroup S of finite index in G. Let π be a finite subset of $\pi(S)$ such that $p, q \notin (\pi' \cap \pi(S))$. By Lemma 2.5.13 of [?], $G/O_{\pi'}(S)$ is a Chernikov group and hence $O_{\pi'}(S)$ has infinite rank. Zaicev's Theorem (see [?]) guarantees that there is an abelian subgroup $B = B_1 \times B_2$ in $O_{\pi'}(S)$ such that B_1 and B_2 have infinite rank and both are normalized by x. Then the q'-subgroups $B_i\langle x \rangle$ has infinite rank and therefore $(B_i\langle x \rangle)Q$ is a subgroup of G.
So

$$\langle x \rangle Q = B_1\langle x \rangle Q \cap B_2\langle x \rangle Q$$

is a subgroup of G. In particular, $\langle x \rangle$ is S-semipermutable in G and then every subgroup of G is S-semipermutable by Lemma 2. □

Theorem B cannot be extended to arbitrary periodic soluble groups, which does not satisy the minimal condition on p-subgroups for every prime number p.

Proposition. *There exists a metabelian group G with infinite rank whose subgroup of infinite rank are S-semipermutable but not every subgroup of G is S-semipermutable*

Proof. For every integer $i \geq 1$, let

$$S_i = \langle a_i, b_i \mid a_i^3 = b_i^2 = 1, b_i^{-1}a_ib_i = a_i^{-1} \rangle$$

be an isomorphic copy of the symmetric group on three letters S_3 and let $S = \mathrm{Dr}_{i\geq 1} S_i$.
Let $P = \mathrm{Dr}_{i\geq 1}\langle b_i \rangle$ and let $Q = \langle a_1 \rangle \times \langle a_2 \rangle$ and consider $G = PQ$. Observe that P is a 2-elementary abelian group of infinite rank, so that G is a countable metabelian group of infinite rank.
Let A be a subgroup of G of infinite rank. Since G has a finite normal Sylow q-subgroup, there are only two possibilities for the set $\pi(A)$: either $\pi(A) = \{2, 3\}$ or $\pi(A) = \{2\}$. In the first case, A is trivially S-semipermutable in G. In the second case, A permutes with the normal Sylow q-subgroup Q. Then every subgroup of G of infinite rank is S-semipermutable.
By contradiction, suppose that every subgroup of G is S-semipermutable. Let $X = \langle a_1a_2 \rangle$. X is S-semipermutable and then PX is a subgroup of G. Since $X = PX \cap Q$, X is a normal subgroup of PX but this is a contradiction since the element

$$b_1^{-1}a_1a_2b_1 = b_1^{-1}a_1b_1a_2 = a_1^2a_2$$

does not belong to X. □

References

[1] **M.R. Dixon** *Sylow theory, Formations and Fitting classes in Locally Finite Groups*, World Scientific (1994).

[2] **B. Hartley** *Finite groups of automorphisms of locally soluble groups*, Journal of Algebra 57, 242-251 (1979).

Intense automorphisms of p-groups

Mima Stanojkovski, Algant PhD student supervised by Prof. Hendrik Lenstra and Prof. Andrea Lucchini
with contributions by Jon González Sánchez (EHU)
Universiteit Leiden and Università degli Studi di Padova

Intense automorphisms

Let G be a group. An automorphism α of G is *intense* if for all subgroups H of G there exists $g \in G$ such that $\alpha(H) = gHg^{-1}$. Denote by $\mathrm{Int}(G)$ the collection of intense automorphisms of G; then $\mathrm{Int}(G) \lhd \mathrm{Aut}(G)$.

Examples: Inner automorphisms are intense. If V is a vector space over a prime field $\mathbb{F}_p$, then the intense automorphisms of V are the scalar multiplications by elements of $\mathbb{F}_p^*$.

Equivalence relation: Let G, G' be groups and let α, β be intense automorphisms respectively of G and G'. The pairs (G, α) and (G', β) are *equivalent* if there exists an isomorphism $\sigma : G \to G'$ such that $\beta\sigma = \sigma\alpha$.

The general setting

Let G be a finite group. A lot can be said about the structure of G once the structure of $\mathrm{Aut}(G)$ is known. Besides, in some cases, very few assumptions on $\mathrm{Aut}(G)$ can lead to very strong limitations to the shape of G.

Intense automorphisms are a generalization of power automorphisms and, in some sense, they resemble class-preserving automorphisms. If G is a non-abelian p-group then both power and class-preserving automorphisms have order equal to a power of p, but the same need not hold for the elements of $\mathrm{Int}(G)$. We will explore this last situation extensively and see how intense automorphisms give rise to a (surprisingly!) rich theory.

The case of p-groups

Let p be a prime number and let G be a finite p-group. Then $\mathrm{Int}(G) = P_G \rtimes C_G$, where P_G is the unique Sylow p-subgroup of $\mathrm{Int}(G)$ and C_G is a cyclic group of order dividing $p-1$. The *intensity* of G is $\mathrm{int}(G) = \#C_G$.

Goal: We want to understand finite p-groups G whose group of intense automorphisms $\mathrm{Int}(G)$ is not itself a p-group. In other words, we want to know when $\mathrm{int}(G) > 1$. *As this can never happen for 2-groups, we will only be working with odd primes.*

Strategy: Let $\mathcal{T}_p$ be the collection of equivalence classes of pairs (G, α) such that G is a finite p-group and α is conjugate to a *non-trivial* element of C_G. For all $c \in \mathbb{Z}_{\geq 0}$, define

$$\mathcal{T}_p[c] = \{\, [G, \alpha] \in \mathcal{T}_p \ :\ G \text{ has class } c \,\}$$

and note that the collection $\{\, \mathcal{T}_p[c] \,\}_{c \geq 1}$ is a partition of $\mathcal{T}_p$.

Small nilpotency classes

Let p be an odd prime. Then the following hold.

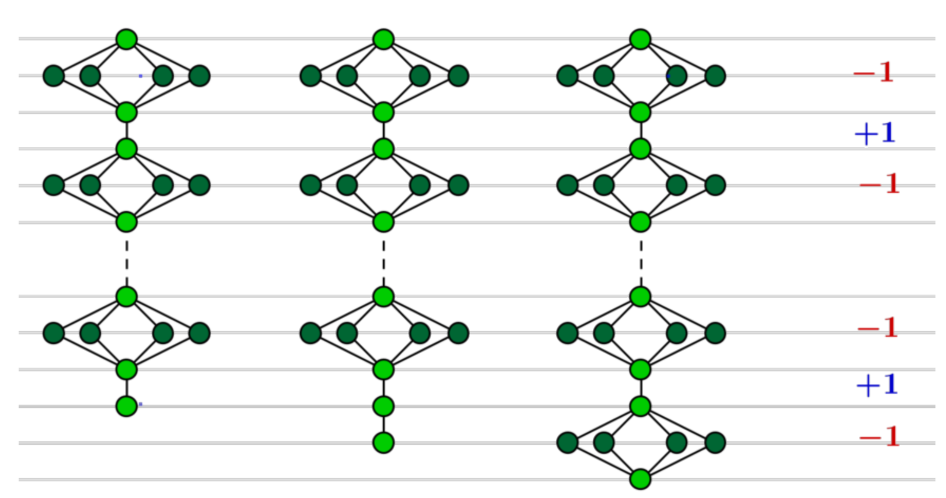
1. $\mathcal{T}_p[1] = \{\, [G, \alpha] \ :\ G \neq 1 \text{ abelian},\ \alpha \in \omega(\mathbb{F}_p^*) \setminus \{1\} \,\}$, where $\omega : \mathbb{F}_p^* \to \mathbb{Z}_p^*$ is the Teichmüller character.
2. $\mathcal{T}_p[2] = \{\, [\mathrm{ES}_p(n), \alpha_\lambda] \ :\ n \in \mathbb{Z}_{\geq 1}, \lambda \in \mathbb{F}_p^* \setminus \{1\} \,\}$, where $\mathrm{ES}_p(n)$ is extraspecial, of order p^{2n+1} and exponent p, and α_λ is a lift of λ-th powering on $\mathrm{ES}_p(n)/\Phi(\mathrm{ES}_p(n))$.

Note: If G is a finite p-group of class at most 2, then $\mathrm{int}(G)$ is either 1 or $p-1$. Moreover, both $\mathcal{T}_p[1]$ and $\mathcal{T}_p[2]$ are infinite.

Higher nilpotency classes

Let p be an odd prime. Let $c \geq 3$ and let $[G, \alpha] \in \mathcal{T}_p[c]$. Then the following hold.

1. The order of α is equal to 2 and $\mathrm{int}(G) = 2$.
2. The lower central series and p-central series of G coincide.
3. The map α induces the inversion map on $G/\Phi(G)$.
4. The group G is thin, with one of the following diagrams.



Theorem

Let p be an odd prime and let $c \in \mathbb{Z}_{>0}$. Then the following hold.

1. If $c \geq 3$, then $\mathcal{T}_p[c]$ is finite.
2. $\mathcal{T}_p[c] = \emptyset \iff p = 3$ and $c \geq 5$.
3. The set $\mathcal{T}_3[4]$ has exactly one element.
4. If $p > 3$, then $\#\varprojlim_c \mathcal{T}_p[c] = 1$.

If $\varprojlim_c \mathcal{T}_p[c] = \{[G^{(c)}, \alpha^{(c)}]\}_{c>0}$, we want to determine the pro-p-group $G_{\lim} = \varprojlim_c G^{(c)}$ and the automorphism $\alpha_{\lim}$ of $G_{\lim}$ that is induced by the automorphisms $\alpha^{(c)}$.

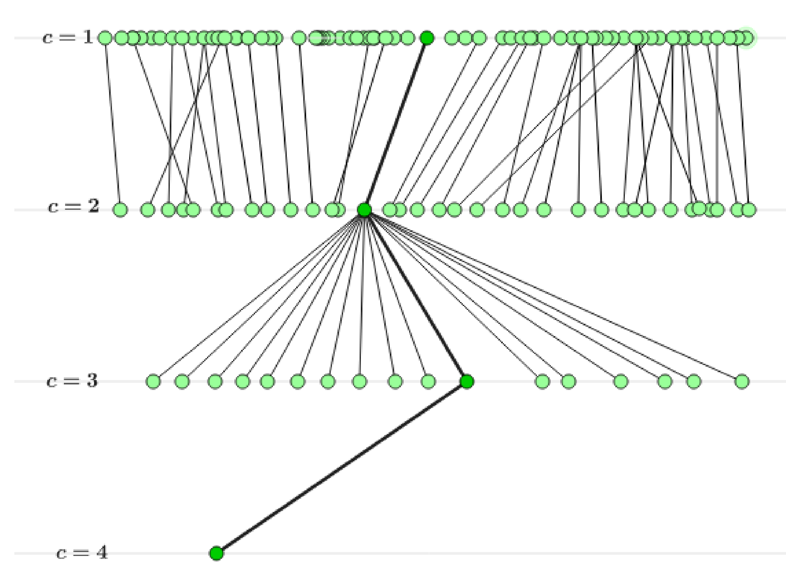
INTENSE PROJECTIVE SYSTEM (IPS)

There is a well-defined sequence of sets

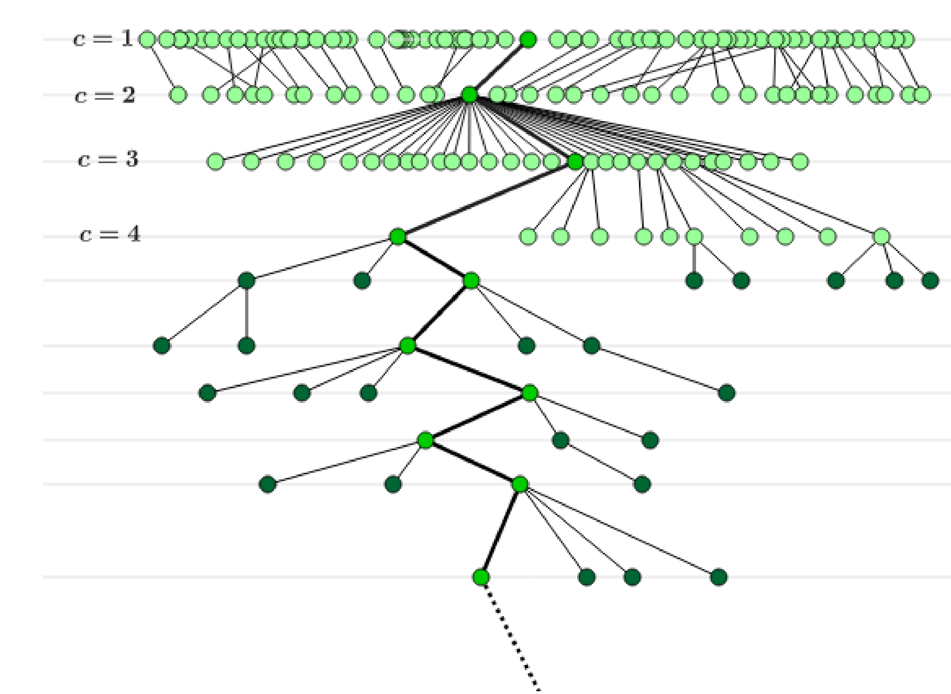
$$\ldots \longrightarrow \mathcal{T}_p[c+1] \xrightarrow{\pi_{c+1}} \mathcal{T}_p[c] \xrightarrow{\pi_c} \mathcal{T}_p[c-1] \longrightarrow \quad \ldots \quad \longrightarrow \mathcal{T}_p[1]$$

where, for all c, the map π_c is defined by $\pi_c : [G, \alpha] \mapsto [G/\gamma_c(G), \overline{\alpha}]$.
The sequence $(\gamma_i(G))_{i \geq 1}$ denotes the lower central series of G and $\overline{\alpha}$ is the map induced by α on $G/\gamma_c(G)$.

IPS for $p = 3$



IPS for $p \geq 5$



A maximal class example

Let $k = \mathbb{F}_3[\epsilon]$, where $\epsilon^2 = 0$, and set $\mathrm{A}_3 = k + k\mathrm{i} + k\mathrm{j} + k\mathrm{ij}$, where i, j satisfy $\mathrm{i}^2 = \mathrm{j}^2 = \epsilon$, and $\mathrm{ji} = -\mathrm{ij}$. The quaternion algebra A_3 is local, with maximal ideal $\mathfrak{m} = \mathrm{A}_3\mathrm{i} + \mathrm{A}_3\mathrm{j}$ and canonical anti-homomorphism

$$a = s + t\mathrm{i} + u\mathrm{j} + v\mathrm{ij} \ \mapsto\ \overline{a} = s - t\mathrm{i} - u\mathrm{j} - v\mathrm{ij}.$$

Let $G_{\max} = \{a \in 1 + \mathfrak{m} \ :\ a\overline{a} = 1\}$ and let the automorphism $\alpha_{\max} : G_{\max} \to G_{\max}$ be defined by

$$a = s + t\mathrm{i} + u\mathrm{j} + v\mathrm{ij} \ \mapsto\ \alpha_{\max}(a) = s - t\mathrm{i} - u\mathrm{j} + v\mathrm{ij}.$$

Fact: $\mathcal{T}_3[4] = \{[\,G_{\max}, \alpha_{\max}\,]\}$.

Another construction

Let J_3 denote the third Janko group and let $3.\mathrm{J}_3$ denote its Schur cover. Let S be a Sylow 3-subgroup of $3.\mathrm{J}_3$ and let N be its normalizer. Let x be an element of order 2 in N and let $\iota_x : S \to S$ be conjugation under x.

Fact: $\mathcal{T}_3[4] = \{[S, \iota_x]\}$.

Thanks to: Derek Holt and Frieder Ladisch for this characterization.

A profinite example

Let $p > 3$ be a prime and let $t \in \mathbb{Z}_p$ satisfy $\left(\frac{t}{p}\right) = -1$. Set $\mathrm{A}_p = \mathbb{Z}_p + \mathbb{Z}_p\mathrm{i} + \mathbb{Z}_p\mathrm{j} + \mathbb{Z}_p\mathrm{ij}$ with defining relations $\mathrm{i}^2 = t$, $\mathrm{j}^2 = p$, and $\mathrm{ji} = -\mathrm{ij}$. Then A_p is a non-commutative local ring such that $\mathrm{A}_p/\mathrm{jA}_p \cong \mathbb{F}_{p^2}$. The involution $\overline{\cdot} : \mathrm{A}_p \to \mathrm{A}_p$ is defined by

$$a = s + t\mathrm{i} + u\mathrm{j} + v\mathrm{ij} \ \mapsto\ \overline{a} = s - t\mathrm{i} - u\mathrm{j} - v\mathrm{ij}.$$

Let $\mathrm{SL}(p) = \{a \in 1 + \mathrm{jA}_p \ :\ a\overline{a} = 1\}$ and let α_p be the automorphism of $\mathrm{SL}(p)$ that is defined by

$$a = s + t\mathrm{i} + u\mathrm{j} + v\mathrm{ij} \ \mapsto\ \alpha_p(a) = s + t\mathrm{i} - u\mathrm{j} - v\mathrm{ij}.$$

Theorem

The group $\mathrm{SL}(p)$ is a pro-p-group and α_p is topologically intense, i.e. for any closed subgroup H of $\mathrm{SL}(p)$ there exists $g \in \mathrm{SL}(p)$ such that $\alpha_p(H) = gHg^{-1}$. Moreover, $(\,\mathrm{SL}(p), \alpha_p\,) \cong (\,G_{\lim}, \alpha_{\lim}\,)$.

Generating transitive permutation groups

Gareth Tracey
Mathematics Institute, University of Warwick
G.M.Tracey@warwick.ac.uk

How many elements does it take to generate a minimally transitive permutation group?

Consider the following question: what is the smallest number $f(n)$ we can find, such that for any transitive permutation group G of degree n, one can find an $f(n)$-generated subgroup of G which is also transitive?

Definition 1 *A transitive permutation group G is called* minimally transitive *if every proper subgroup of G is intransitive.*

For a prime factorisation $n = \prod_{p \text{ prime}} p^{n(p)}$ of n, set $\omega(n) := \sum_{p \text{ prime}} n(p)$ and $\mu(n) := \max\{n(p) : p \text{ prime}\}$.

In the language of Definition 5, our task is to find the best possible upper bound on $d(G)$, in terms of n, where G is a minimally transitive group of degree n. The question was first considered by Shepperd and Wiegold:

Theorem 2 (Shepperd; Wiegold, 1963 (CFSG)) *Let G be a minimally transitive permutation group of degree n. Then $d(G) \leq \omega(n)$.*

The statement of the theorem of Shepperd and Wiegold included the hypothesis that every finite simple group can be generated by 2 elements. Of course, as a result of the CFSG, we know that this hypothesis holds true. However, it meant that the result could not be used in general, and this led Neumann and Vaughan-Lee to prove the following:

Theorem 3 (Neumann; Vaughan-Lee, 1977) *Let G be a minimally transitive permutation group of degree n. Then $d(G) \leq \log_2 n$.*

A conjecture was then made, on the bound one should aim for:

Conjecture 4 (Pyber, 1991) *Let G be a minimally transitive permutation group of degree n. Then $d(G) \leq \mu(n) + 1$.*

The conjecture was verified by Pyber himself in the nilpotent case, and then by Lucchini in the soluble case:

Theorem 5 (Lucchini, 1996) *Let G be a soluble minimally transitive permutation group of degree n. Then $d(G) \leq \mu(n) + 1$.*

Finally, we can offer a complete solution to the problem.

Theorem 6 (T., 2015 (CFSG)) *Let G be a minimally transitive permutation group of degree n. Then $d(G) \leq \mu(n) + 1$.*

How many elements does it take to generate a transitive permutation group?

The problem of bounding $d(G)$, for a transitive permutation group G, in terms of its degree, was first considered by McIver and Neumann:

Theorem 7 (McIver; Neumann, 1987) *Let G be a transitive permutation group of degree $n \geq 5$. Then $d(G) < n/2$, except that $d(G) = 4$ when $n = 8$ and $G \cong D_8 \circ D_8$.*

However, it was long suspected that substantially tighter bounds could be proved.

Theorem 8 (Lucchini, 2000 (CFSG)) *Let G be a transitive permutation group of degree $n \geq 2$. Then $d(G) = O(n/\sqrt{\log_2 n})$.*

Note that the constant involved in the previous theorem was never estimated. In fact, until 2015 Neumann's 1989 result was the best numerical bound we had for $d(G)$ in terms of n. We now have the following:

Theorem 9 (T., 2015 (CFSG)) *Let G be a transitive permutation group of degree $n \geq 2$. Then $d(G) \leq \left\lfloor \frac{cn}{\sqrt{\log_2 n}} \right\rfloor$ where $c = \sqrt{3}/2 = 0.866025\ldots$, apart from a finite list of exceptions.*

Theorem 10 (T., 2016 (CFSG)) *Let G be a transitive permutation group of degree $n \geq 2$. Then*

$$d(G) = O\left(\frac{n^2}{\log|G|\sqrt{\log_2 n}}\right)$$

References

1. W. M. Kantor, A. Lubotzky and A. Shalev, Invariable generation and the Chebotarev invariant of a finite group, J. Algebra 348 (2011), 302–314.

How many elements does it take to invariably generate a permutation group?

Definition 11 *A subset $\{x_1, x_2, \ldots, x_t\}$ of a group G is said to* invariably generate *G if $\langle x_1^{g_1}, x_2^{g_2}, \ldots, x_t^{g_t}\rangle = G$ for every t-tuple $(g_1, g_2, \ldots, g_t)$ of elements of G. The cardinality of the smallest invariable generating set for G is denoted by $d_I(G)$.*

Several recent papers have discussed upper bounds on $d_I(G)$ for a finite group G. Clearly $d_I(G)$ is at least $d(G)$, but how large is the difference $d_I(G) - d(G)$? In general, the answer is: arbitrarily large (see [1, Proposition 2.5]). One can, however, consider a related question: Suppose that for a class $\mathcal{C}$ of finite groups, we have an upper bound on $d(G)$, for G in $\mathcal{C}$, in terms of some invariant of $\mathcal{C}$. Does said upper bound still hold if one replaces d by d_I?

For instance, the case when $\mathcal{C}$ is the set of permutation groups of degree n was considered by Detomi and Lucchini:

Theorem 12 (Detomi; Lucchini, 2014 (CFSG)) *Let G be a permutation group of degree n. Then $d_I(G) \leq \lfloor \frac{n}{2} \rfloor$, except that $d_I(G) = 2$ when $n = 3$ and $G \cong \mathrm{Sym}(3)$.*

We can now also prove the following:

Theorem 13 (T., 2015 (CFSG)) *Let G be a transitive permutation group of degree n. Then $d_I(G) = O(n/\sqrt{\log n})$.*

Theorem 14 (T., 2015 (CFSG)) *Let G be a primitive permutation group of degree n. Then $d_I(G) = O(\log n/\sqrt{\log\log n})$.*

Characterisations of groups in which normality is a transitive relation by means of subgroup embedding properties

Ramón Esteban-Romero (Universitat de València/Universitat Politècnica de València, Ramon.Esteban@uv.es) and Giovanni Vincenzi (Università degli Studi di Salerno, vincenzi@unisa.it), Ischia Group Theory 2016

	Finite groups	**FC*-groups**	**Groups without infinite simple sections**
A group G is said to be: 1. a *T-group* if $H \trianglelefteq K \trianglelefteq G$ implies $H \trianglelefteq G$; 2. a *$\bar{T}$-group* if every subgroup of G is a T-group; 3. *locally graded* if every non-trivial finitely generated subgroup of G has a non-trivial finite homomorphic image; 4. an *FC^0-group* if G is finite and, by induction, an *FC^{n+1}-group* if $G/\mathrm{C}_G(\langle x\rangle^G)$ is an FC^n-group for all $x \in G$; then G is an FC*-group if G is an FC^n-group for some $n \geq 0$.	Let G be a finite group. 1. If G is a soluble T-group, then G is a $\bar{T}$-group. 2. If G is a $\bar{T}$-group, then G is soluble. [Gaschütz, 1957], [Zacher, 1952]) Examples of infinite soluble T-groups that are not $\bar{T}$-groups are constructed in [Robinson, 1964] and [Kuzennyi and Subbotin, 1989].	Let G be a soluble FC*- group. Then the following statements are equivalent: 1. G is a T-group. 2. G is a $\bar{T}$-group. [Esteban-Romero and Vincenzi, 2016, Theorem 2.3]	1. Let G be a group without infinite simple sections. Then: (a) G is locally graded. (b) If G is a $\bar{T}$-group, then G is metabelian. 2. Let G be a soluble group. Then G is a $\bar{T}$-group $\iff$ every ascendant subgroup of G is normal in G. [de Giovanni and Vincenzi, 2000, Theorem 3.6] **Open question** (see [Mazurov and Khukhro, 2014, Question 14.36]). Are non-periodic locally graded $\bar{T}$-groups soluble?
A subgroup X of a group G is said to be *pseudonormal* [de Giovanni and Vincenzi, 2003] or *transitively normal* [Kurdachenko and Subbotin, 2006] or *to satisfy the subnormaliser condition* [Mysovskikh, 1999] if $\mathrm{N}_G(H) \leq \mathrm{N}_G(X)$, for each subgroup H of G such that $X \leq H \leq \mathrm{N}_G(X)$. This is equivalent to affirming that if $H \leq L$ and H is subnormal in L, then $H \trianglelefteq L$.	Let G be a finite group, then the following are equivalent: 1. G is a $\bar{T}$-group. 2. Every subgroup of G is pseudonormal. [Ballester-Bolinches and Esteban-Romero, 2003, Theorem A]	Let G be an FC*-group, then the following are equivalent: 1. G is a $\bar{T}$-group 2. Every subgroup of G is pronormal 3. Every subgroup of G is pseudonormal It follows by [de Giovanni and Vincenzi, 2003, Theorem 3.1 and Corollary 3.5] and [de Giovanni et al., 2002, Theorem 4.6] or [Romano and Vincenzi, 2011, Theorem 3.3]	A group G is a $\bar{T}$-group if and only if all its subgroup are pseudonormal [de Giovanni and Vincenzi, 2003, Theorem 3.1]
A subgroup X of a group G is said to be *pronormal* if X and X^g are conjugate in $\langle X, X^g\rangle$, for every element $g \in G$.	Let G be a finite soluble group. Then the following are equivalent: 1. G is a T-group. 2. G is a $\bar{T}$-group. 3. X is pronormal in G, $\forall X \leq G$. [Peng, 1969]	Let G be a soluble FC*- group. Then the following are equivalent: 1. G is a T-group. 2. X is pronormal in G for all $X \leq G$. [de Giovanni and Vincenzi, 2000, Theorem 3.9]	Let G be a group without infinite simple sections. Then G is a $\bar{T}$-group $\iff$ every cyclic subgroup is pronormal Examples of $\bar{T}$-groups containing non pronormal subgroups were given by [Kovács et al., 1961], and by [Kuzennyi and Subbotin, 1989].
A subgroup X of a group G is said to be *weakly normal* [Müller, 1966] if $X^g \leq \mathrm{N}_G(H)$ implies $g \in \mathrm{N}_G(H)$	Let G be a finite soluble group. Then the following are equivalent: 1. G is a T-group. 2. Every subgroup of G is weaky normal. [Ballester-Bolinches and Esteban-Romero, 2003, Theorem A]	$\rightarrow$	Let G be a group. Then the following are equivalent: 1. G is a $\bar{T}$-group without infinite simple sections. 2. G is a localy graded group whose subgroup are weaky normal. [Russo, 2012, Corollary 4], [Romano and Vincenzi, 2015, Theorem 2.8]
A subgroup X of a group G is said to be an *$\mathcal{H}$-subgroup* or that it has the *$\mathcal{H}$-property* in G if $\mathrm{N}_G(X) \cap X^g \leq X$ for all elements g of G.	Let G be a finite soluble group. Then the following are equivalent: 1. G is a T-group. 2. G is a $\bar{T}$-group. 3. Every subgroup of G has the property $\mathcal{H}$. [Bianchi et al., 2000, Theorem 10]	$\rightarrow$	Let G be a group without infinite simple sections. Then the following are equivalent: 1. G is a $\bar{T}$-group. 2. Every subgroup of G has the property $\mathcal{H}$. [Vincenzi, 2016, Theorem 3.2]
A group G is said to be an *NNM-group* (non-normal maximal) if each non-normal subgroup of G is contained in a non-normal maximal subgroup of G.	Let G be a finite soluble group. Then the following are equivalent: 1. G is a T-group. 2. All subgroups of G are NNM-groups. [Kaplan, 2011b, Theorem 1]	Let G be a soluble FC*-group. Then the following are equivalent: 1. G is a soluble T-group. 2. All subgroups of G are NNM-groups. [Esteban-Romero and Vincenzi, 2016, Theorem 2.5]	There exist examples of T-groups, that are hyperfinite and FC-nilpotent but that are not NNM-groups [Esteban-Romero and Vincenzi, 2016, Example 2.6].
A subgroup H of a group G is said to be a *φ-subgroup* of G if, for all K, L maximal in H, it is the case that if K, L are conjugate in G, then K, L are conjugate in H. A subgroup K of a group G is said to be a *cr-subgroup* (conjugation restricted) of G if there are no $A < K$, $g \in G$ such that $K = AA^g$.	Let G be a finite soluble group. Then the following are equivalent: 1. G is a T-group. 2. Every subgroup of G has the property φ. 3. Every subgroup of G is a cr-subgroup. [Kaplan, 2011a, Theorem 7]	Let G be a soluble FC*-group. Then the following are equivalent: 1. G is a T-group. 2. Every subgroup of G has the property φ. 3. Every subgroup of G is a cr-subgroup. [Kaplan and Vincenzi, 2014, Theorem 5.2]	To investigate
A subgroup H of a finite group G is called an *NE-subgroup* [Li, 1998] if it satisfies $\mathrm{N}_G(H) \cap H^G = H$.	Let G be a finite soluble group. Then the following are equivalent: 1. G is a T-group; 2. Every subgroup of G is an NE-subgroup of G. [Li, 2006, Theorem 3.1]	$\rightarrow$	Let G be a group without infinite simple sections. Then the following are equivalent: 1. G is a soluble $\bar{T}$-group. 2. Every subgroup of G is an NE-subgroup of G. [Esteban-Romero and Vincenzi, in progress]

References

[Ballester-Bolinches and Esteban-Romero, 2003] Ballester-Bolinches, A. and Esteban-Romero, R. (2003). On finite $\mathcal{T}$-groups. *J. Austral. Math. Soc. Ser. A*, 75:1–11.

[Bianchi et al., 2000] Bianchi, M., Mauri, A. G. B., Herzog, M., and Verardi, L. (2000). On finite solvable groups in which normality is a transitive relation. *J. Group Theory*, 3(2):147–156.

[de Giovanni et al., 2002] de Giovanni, F., Russo, A., and Vincenzi, G. (2002). Groups with restricted conjugacy classes. *Serdica Math. J.*, 28(3):241–254.

[de Giovanni and Vincenzi, 2000] de Giovanni, F. and Vincenzi, G. (2000). Pronormality in infinite groups. *Math. Proc. R. Ir. Acad.*, 100A(2):189–203.

[de Giovanni and Vincenzi, 2003] de Giovanni, F. and Vincenzi, G. (2003). Pseudonormal subgroups of groups. *Ricerche Mat.*, 52(1):91–101.

[Esteban-Romero and Vincenzi, 2016] Esteban-Romero, R. and Vincenzi, G. (2016). On generalised FC-groups in which normality is a transitive relation. *J. Austral. Math. Soc. Ser. A*, 100(2):192–198.

[Gaschütz, 1957] Gaschütz, W. (1957). Gruppen, in denen das Normalteilersein transitiv ist. *J. Reine Angew. Math.*, 198:87–92.

[Kaplan, 2011a] Kaplan, G. (2011a). On finite T-groups and the Wielandt subgroup. *J. Group Theory*, 14:855–863.

[Kaplan, 2011b] Kaplan, G. (2011b). On T-groups, supersolvable groups, and maximal subgroups. *Arch. Math. (Basel)*, 96(1):19–25.

[Kaplan and Vincenzi, 2014] Kaplan, G. and Vincenzi, G. (2014). On the Wielandt subgroup of generalized FC-groups. *Int. J. Algebra Comput.*, 24(7):1031–1042.

[Kovács et al., 1961] Kovács, L. G., Neumann, B. H., and de Vries, H. (1961). Some Sylow subgroups. *Proc. Roy. Soc. Ser. A*, 260:304–316.

[Kurdachenko and Subbotin, 2006] Kurdachenko, L. A. and Subbotin, I. Y. (2006). Transitivity of normality and pronormal subgroups. In Fine, B. et al., editors, *Combinatorial group theory, discrete groups, and number theory. A conference in honor of Gerhard Rosenberger's sixtieth birthday, Fairfield, CT, USA, December 8–9, 2004 and the AMS special session on infinite groups, Bard College, Annandale-on-Hudson, NY, USA, October 8–9, 2005*, volume 421 of *Contemporary Mathematics*, pages 201–212, Providence, RI. American Mathematical Society (AMS).

[Kuzennyi and Subbotin, 1989] Kuzennyi, N. F. and Subbotin, I. Y. (1989). Groups with pronormal primary subgroups. *Ukr. Math. J.*, 41:286–289.

[Li, 1998] Li, S. (1998). On minimal non-PE-groups. *J. Pure Appl. Algebra*, 132:149–158.

[Li, 2006] Li, Y. (2006). Finite groups with NE-subgroups. *J. Group Theory*, 9:49–58. 2006.

[Mazurov and Khukhro, 2014] Mazurov, V. D. and Khukhro, E. I., editors (2014). *Unsolved problems in Group Theory: The Kourovka Notebook*. Russian Academy of Sciences, Siberian Branch, Institute of Mathematics, Novosibirsk, Russia, 18 edition.

[Müller, 1966] Müller, K. H. (1966). Schwachnormale Untergruppen: Eine gemeinsame Verallgemeinerung der normalen und normalisatorgleichen Untergruppen. *Rend. Sem. Mat. Univ. Padova*, 36:129–157.

[Mysovskikh, 1999] Mysovskikh, V. I. (1999). Investigation of subgroup embeddings by the computer algebra package GAP. In *Computer algebra in scientific computing—CASC'99 (Munich)*, pages 309–315, Berlin. Springer.

[Peng, 1969] Peng, T. A. (1969). Finite groups with pronormal subgroups. *Proc. Amer. Math. Soc.*, 20:232–234.

[Robinson, 1964] Robinson, D. J. S. (1964). Groups in which normality is a transitive relation. *Math. Proc. Camb. Phil. Soc.*, 60:21–38.

[Romano and Vincenzi, 2011] Romano, E. and Vincenzi, G. (2011). Pronormality in generalized FC-groups. *Bull. Austral. Math. Soc.*, 83:220–230.

[Romano and Vincenzi, 2015] Romano, E. and Vincenzi, G. (2015). Groups in which normality is a weakly transitive relation. *J. Algebra Appl.*, 14(1):1550007 (12 pages).

[Russo, 2012] Russo, A. (2012). On groups in which normality is a transitive relation. *Comm. Algebra*, 40:3950–3954.

[Vincenzi, 2016] Vincenzi, G. (in press 2016). A characterization of soluble groups in which normality is a transitive relation. *Int. J. Group Th.*.

[Zacher, 1952] Zacher, G. (1952). Caratterizzazione dei t-gruppi finiti risolubili. *Ricerche Mat.*, 1:287–294.

Acknowledgements

The research of the first author has been supported by the research grant MTM2014-54707-C3-1-P by the *Ministerio de Economía y Competitividad* (Spain) and FEDER (European Union). The research of the second author has been supported by the GNSAGA-INdAM *Ministero dell'Università e della ricerca* (Italy).