On Left 3-Engel elements

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1. Some general background.
2. Left 3-Engel elements in groups.
3. Groups of exponent 5.

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The converse does not hold in general (Golod's examples). But it does for certain classes like groups satisfying max (Baer) and solvable groups (Gruenberg).

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If the left 4-Engel elements are in HP(G) then groups of exponent 8 are locally finite.

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TFAE: (1) Left 3 Engel elements are always contained in the locally nilpotent radical.
(2) Every sandwich group is locally nilpotent.

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Definition.A 3-generator group $\langle a, b, c\rangle$ is pairwise nilpotent of type (r, s, t) if $\langle a, b\rangle$, $\langle a, c\rangle$ and $\langle b, c\rangle$ are nilpotent of class r, s and t respectively.

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