Commutator width of Chevalley groups

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Theorem (Ore'51, Ellers—Gordeev'98, Liebeck—O'Brien—Shalev—Tiep'10)

Every element of a non-abelian finite simple group is a commutator.

Theorem (Ree'64)

Every element of a connected semisimple algebraic group over an algebraically closed field is a commutator.

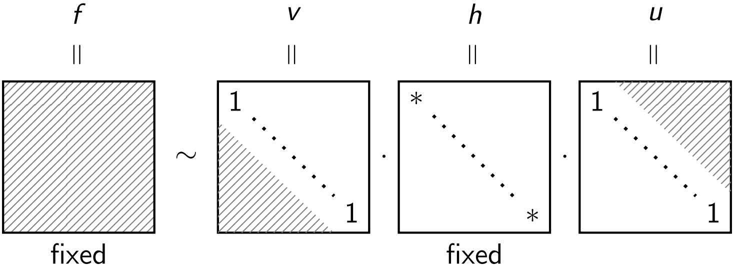
Question

What about linear groups over rings?

Gauss decomposition with prescribed semisimple part

Theorem (Ellers—Gordeev'94–96)

Let K be a field with more than 8 elements and G an almost simple simply connected algebraic group, defined and split over K. Then for any non-central $f \in G$ and any $h \in T$ one has $f \sim vhu$, where $v \in U^-$ and $u \in U^+$.



Gauss decomposition and unitriangular factorization

Definition

Commutative ring R is of stable rank 1 if for any $a, b \in R$ such that $aR + bR = R$ there exists $c \in R$ with $a + bc \in R^*$.

Theorem

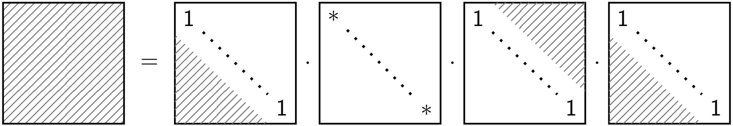
For a commutative ring R of stable rank 1 and a root system Φ the elementary Chevalley group $E(\Phi, R)$ admits the following two decompositions:

$$E(\Phi, R) = U^+ T U^- U^+ \text{ (Gauss decomposition)},$$

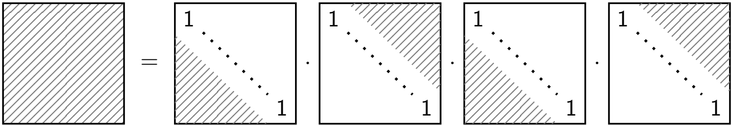
$$E(\Phi, R) = U^+ U^- U^+ U^- \text{ (unitriangular factorization)}.$$

Gauss decomposition and unitriangular factorization

Gauss decomposition:



Unitriangular factorization:



Theorem (Vaserstein—Wheland'90)

For a ring R of stable rank 1 every element of $E(n, R)$ is a product of at most 2 commutators of elements from $GL(n, A)$.

Theorem (Arlinghaus—Vaserstein—You'95)

For a form ring (R, Λ) of Λ-stable rank 1 every element of the elementary hyperbolic unitary group $EU(2n, R, \Lambda)$ is a product of at most 4 commutators from $EU(2n, R, \Lambda)$ and a product of at most 3 commutators from $GU(2n, R, \Lambda)$.

Commutator width of Chevalley groups

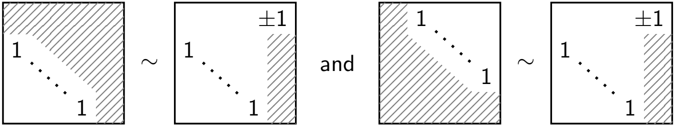
Theorem

For a commutative ring R of stable rank 1 and a root system Φ every element of the elementary Chevalley group $E(\Phi, R)$ is a product of at most N commutators from $E(\Phi, R)$, where

- $N = 3$ *in case* $\Phi = \mathrm{A}_\ell, \mathrm{F}_4, \mathrm{G}_2$;
- $N = 4$ *in case* $\Phi = \mathrm{B}_\ell, \mathrm{C}_\ell, \mathrm{D}_\ell, \mathrm{E}_7, \mathrm{E}_8$;
- $N = 5$ *in case* $\Phi = \mathrm{E}_6$.

In the same setting every element of $E(\Phi, R)$ is a product of at most $N - 1$ commutators from $\widetilde{G}(\Phi, R)$.

Sketch of the proof

1) 

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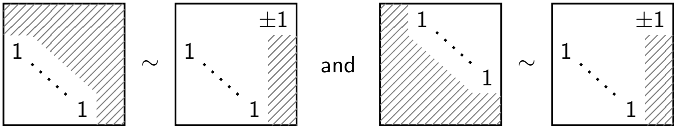

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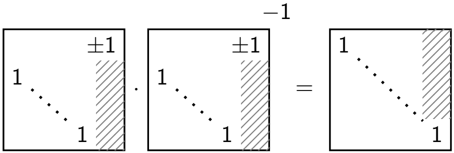

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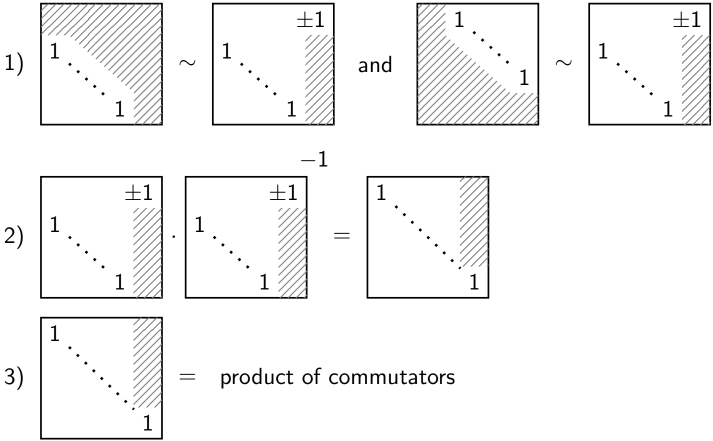

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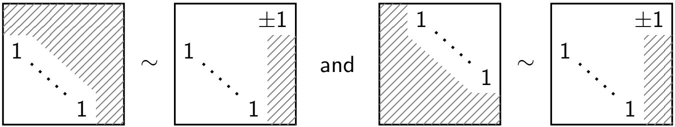
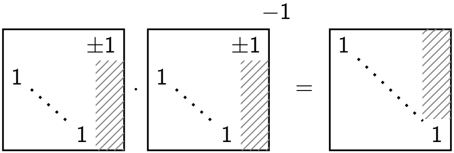
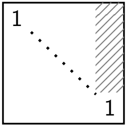
1) 

2) 

3) 

= product of commutators

4) apply all of the above to $E(\Phi, R) = U^+U^-U^+U^-$.

Commutator width: nice rings

- For $R = \mathbb{Z}[1/p]$ one has $E(\Phi, R) = U^+U^-U^+U^-U^+$ (Sury—Vsemirnov), so the commutator width is the same as for rings of stable rank 1;

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- For R a boolean ring one has $E(\Phi, R) = U^+U^-U^+$, so $w_C(E(\Phi, R)) = 2$ for $\mathrm{A}_\ell, \mathrm{F}_4, \mathrm{G}_2$, $= 4$ for E_6 and $= 3$ in all other cases;

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- Good estimates can be obtained for the rings of holomorphic functions on Stein manifolds (Ivarsson—Kutzschebauch).

Commutator width: not so nice rings

- $SL(2,\mathbb{Z}) \subset \big(U^+(3,\mathbb{Z})\, U^-(3,\mathbb{Z})\big)^{20}$ by a result of Carter and Keller, and it follows that $SL(n \geqslant 60, \mathbb{Z}) = \big(U^+ U^-\big)^3$, therefore $w_C(SL(n,\mathbb{Z})) \leqslant 4$ for $n \geqslant 60$.
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  The same can be done for other classical groups, but with much worse bounds.
- $SL(n,\mathbb{C}[t])$ does not have finite width with respect to elementary generators (van der Kallen) or commutators (Dennis—Vaserstein);

This is where my talk ends.
Thank you for your attention.