My joint research with the late Professor David Chillag (1946-2012)

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Topic 1: [1],[8],[10] and [12] On finite groups containing a CC-subgroup

Definition

A proper subgroup M of a group G is called a CC-subgroup of G if the centralizer $C_G(m)$ of every $m \in M^{\#} = M \setminus \{1\}$ is contained in M.

In Ischia Group Theory 2004 we edited a proceedings of the Conference in Honor of Marcel Herzog Volume 402 of Contemporary Mathematics.
In this volume I published a paper (jointly with W. Herfort) on "The History of the Classification of Finite Groups with a CC-subgroup".

On finite groups containing a CC-subgroup

In this lecture we stated the full classification of finite groups with a CC-subgroup. The proof was based on 40 papers among them 4 joint papers of Arad and Chillag. The problem was started by Frobenius around 100 years ago.
Many authors contributed to the final solution among them David Chillag, Arad, Marcel Herzog, Wolfgang Herfort, Higman, Feit, P. Hall, Kegel, Grunberg, Suzuki, Carter, Conway, Passman, Wiliams, Lucido, Kondrateiv, Mazurov and others.
To state the classification cc-theorem in vol. 402 of Contemporary Mathematics mentioned above, it took 7 pages so one can find the full detailed information in this volume.

Topic 2: [4] Injectors of finite solvable groups of odd order.

A well known theorem of Wielandt and Kegel state that if G is a finite group and $G = AB$, where A and B are nilpotent subgroups of G then G is solvable. The solvable non-nilpotent group S_3 is a product of two cyclic subgroups. In this example the subgroup of order 3 is normal in S_3. Therefore it is interesting to find a criterion when $G = AB$ is a product of two nilpotent subgroups A and B of a finite group and G is nilpotent. The first result of this type was proven by Arad and Glauberman in 1975. For simplicity I will state a simple version of our result.

Injectors of finite solvable groups of odd order.

Theorem (Arad-Glauberman)

Let G be a finite group of odd order. Assume that $G = AB$, where A is abelian subgroup of G of maximal order and $B \trianglelefteq G$ is nilpotent. Then G is nilpotent.

Glauberman in 1975 based on this result proved that the theorem holds if we assume that A is nilpotent of class at most 2 of maximal order.
Bialostocki in 1975 generalized the result and proved that the Theorem holds if we assume that A is nilpotent of class at most c of maximal order in G for every positive integer $c \in \mathbb{N}$.

Injectors of finite solvable groups of odd order.

In 1979 Arad and Chillag [4] continued the research and proved various results related to this topics.

Definition

Let G be a finite group. Define $J(G)$ to be the subgroup of G generated by all abelian subgroup of G of maximal order. $J(G)$ is called the Thompson subgroup of G.

Definition

Let G be a finite group. A nilpotent subgroup K of G is called an N-injector of G if given any subnormal subgroup H of G, $K \cap H$ is a maximal nilpotent subgroup of H.

This concept is due to B. Fischer who proved that if G is solvable, the N-injectors of G exist and any two of them are conjugate.

Injectors of finite solvable groups of odd order.

Let me mentioned the following results:

Proposition (Arad and Chillag).

Let G be a finite group of odd order. Let us denote by $A_\infty(G)$ the family of nilpotent subgroups of maximal order in G. Then

(i) The set $A_\infty(G)$ is the set of N-injectors of G. In particular, all elements of $A_\infty(G)$ are conjugates.

(ii) $1 \neq ZJ(A) = ZJ(G)$ for each $A \in A_\infty(G)$.

(iii) $F(G) \subseteq A$ for each $A \in A_\infty(G)$. In particular, $\bigcap_{x \in G, A \in A_\infty(G)} A^x = F(G)$.

(iv) If $A, B \in A_\infty(G)$ and assume that A_p and B_q are Sylow p-subgroup and Sylow q subgroup of A and B respectively, where $p, q \mid |A|$, p, q are primes. Then $[A_p, B_q] = 1$.

In 1985 Arad and Herzog published vol. 1112 of Lecture Notes in Mathematic "Products of Conjugacy Classes in Groups". In chapter 1 of this volume we proved:

Theorem [Arad-Herzog-Stavi] (The basic covering theorem)

Let G be a finite nonabelian simple group and let $C \neq 1$ be a conjugacy class in G. Then there exists a positive integer m such that $C^m = G$. Furthermore there exists a positive integer n such that $C^n = G$ for every nontrivial conjugacy class C in G.

We proved that if G has k conjugacy classes, then $C^{\frac{k(k-1)}{2}} = G$ for each non-trivial conjugacy class C of G. We denoted by $cn(G)$ (covering number of G) the minimal values for n such that $C^n = G$ for every non-trivial conjugacy class C of G.

Stavi and Ziser independently proved that
$cn(A_n) = [n/2], n \geq 6$.
Arie Lev proved that $cn(PSL(n, q) = n$ for $n > 3$.
In [13] Arad-Chillag- Moran proved that $cn(Sz(q)) = 3$.
In [13] we proved the following:

Theorem (Arad-Chillag-Moran)

Let G be a finite nonabelian simple group. Then $cn(G) = 2$ iff $G \cong J_1$ the Janko's smallest group.

Today a lot of information is known about the covering numbers of finite nonabelian simple groups. But this topic of research is not the main goal of my lecture.

Let G be a finite nonabelial simple group and θ a complex character of G. Define $Irr(\theta)$ to be the set of irreducible constituents of θ and $Irr(G)$ to be the set of all irreducible characters of G. The character covering number $ccn(G)$ of G is defined as the smallest positive integer in such that $Irr(\chi^m) = Irr(G)$ for all $\chi \in Irr(G)^\#$.
In [14] we study bounds of $ccn(G)$. In particular we proved:

Theorem (Arad-Chillag-Herzog)

Let G be a finite nonabelian simple group then $ccn(G) = 2$ iff $G \cong J_1$.

This result illustrated that there exists an analogy between the theory of products of conjugacy classes of G and the products of irreducible character of G.

In fact Arad and Fisman published an article "An analogy between products of two conjugacy classes and products of two irreducible characters in finite groups " Proc. Of the Edinburgh Math. Soc. 30 (1987), 7-22.
At this point our interest in research splited into two directions.
David generalized the analogy concept to Generalized Circulants and semisimple algebras with positive bases.
My direction was to define (jointly with Harvey Blau) the concept of Table Algebras. The Table Algebras Theory was a generalization and unification of the concepts of products of conjugacy classes and products of irreducible characters in a finite group G. In my lecture I will focus on one classification theorem of specific Table Algebras to illustrate the methods and tools of Table Algebras theory.

Joint papers of Z. Arad and D. Chillag

1. On finite groups containing a cc-subgroup (with D. Chillag), Arch. der Math. 29 (1977), 225-234. 2. On finite groups with conditions on the centralizers of p-elements (with D. Chillag), J. of Algebra 51 (1978), 164-172. 3. On a theorem of N. Ito on factorizable groups (with D. Chillag), Arch. der Math. 30 (1978), 236-239.

4. Injectors of finite solvable groups (with D. Chillag), Comm. in Algebra 7(2), (1979), 115-138.

5. On centralizers of elements of odd order in finite groups (with D. Chillag), J. of Algebra 61 (1979), 269-280.

6. Finite groups with conditions on the centralizers of p-elements (with D. Chillag), Comm. in Algebra 7 (14), (1979), 1447-1468.

Joint papers of Z. Arad and D. Chillag

7. π-solvability and nilpotent Hall subgroups (with D. Chillag), Proc. of Symposia in Pure Math. 37 (1980), 197-199.

8. On a property like Sylows of some finite groups (with D. Chillag), Arch. der Math. 35 (1980), 401-405.

9. Classification of finite groups by a maximal subgroup (with M. Herzog, and D. Chillag), J. of Algebra 71 (1981), 235-244.

10. Finite groups containing a nilpotent Hall subgroup of even order (with D. Chillag), Houston, J. of Math. 7 (1981), 23-32.

Joint papers of Z. Arad and D. Chillag

11. On a problem of Frobenius (with M. Herzog and D. Chillag), J. of Algebra 74 (1982), 516-523.

12. A criterion for the existence of normal π-complements in finite groups (with D. Chillag), J. of Algebra 87 (1984). 472-482.
13. Groups with a small covering number (with D. Chillag and G. Moran), Lecture Notes in Math., Springer-Verlag 1112 (1985), 222-244.
14. Powers of characters of finite groups (with D. Chillag and M. Herzog), J. of Algebra 103 (1986), 241-255.

On Normalized Integral Table Algebras Generated by a Faithful Non-real Element of Degree 3

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Table algebras

Definition

Let $B = \{b_1 = 1, ..., b_k\}$ be a distinuished basis of an associative commutative complex algebra A. A pair (A, B) is called a table algebra if it satisfies the following conditions

1. $b_i b_j = \sum_{m=1}^{k} \lambda_{ijm} b_m$ with λ_{ijm} being non-negative reals;
2. there exists a table algebra automorphism $x \mapsto \bar{x}$ of A whose order divides two such that $\overline{B} = B$ ($^{-}$ defines a permutation on $[1, k]$ via $\overline{b_i} = b_{\bar{i}}$);
3. there exists a coefficient function $g : B \times B \to \mathbb{R}^+$ such that $\lambda_{ijm} = g(b_i, b_m)\lambda_{\bar{j}mi}$

An element b_i is called real if $i = \bar{i}$. For any $x = \sum_{i=1}^{k} x_i b_i$ we set $Irr(x) := \{b_i \in B \,|\, x_i \neq 0\}$.

Table subsets

Definition

A non-empty subset $T \leq B$ is called a table subset if $Irr(T\overline{T}) \subseteq T$. In this case a linear span $S := \langle T \rangle$ of T is a subalgebra of A. The pair (S, T) is called table subalgebra of (A, b).

Faithful elements

Since an intersection of table subsets is a table subset by itself, one can define a table subset generated by an element $b \in B$, notation B_b, as the intersection of all table subsets of B containing b. An element $b \in B$ with $B_b = B$ is called faithful.

Isomorphism between table algebras

Rescaling

Given a table algebra (A, B) one can replace its table basis $B = \{b_1, ..., b_k\}$ by $B' = \{\beta_1 b_1, ..., \beta_k b_k\}$ where β_i's are positive real numbers with $\beta_1 = 1$. A table algebra (A, B') is called a rescaling of (A, B).

Isomorphisms between TA

Two table algebras (A, B) and (A', B') are called isomorphic, notation $(A, B) \cong (A', B')$, if there exists an algebra isomorphism $f : A \to A'$ such that $f(B)$ is a rescaling of B'. In the case of $f(B) = B'$, the algebras are called exactly isomorphic, notation $(A, B) \cong_x (A', B')$.

Degree homomorphism

Theorem (Arad, Blau)

Let (A, B) be a table algebra. Then there exists a unique algebra homomorphism $a \mapsto |a|, a \in A$ onto $\mathbb{C}$ such that $|b| = |\bar{b}| > 0$ holds for all $b \in B$. The number $|b|$ is called the degree of b.

Normalized and standard TAs

An element $b_i \in B$ is called standard (normalized) if $\lambda_{i\bar{i}1} = |b_i|$ $(\lambda_{i\bar{i}1} = 1)$. A table algebra is called standard (normalized) if all the elements of its table basis are standard (normalized). Notice that any table algebra may be rescaled to a standard or normalized one. If (A, B) is normalized, then $g(b_i, b_j) = 1$. For standard table algebras $g(b_i, b_j) = |b_i|/|b_j|$.

The order of a TA

Definiion

The number

$$o(B) := \sum_{i=1}^{k} \frac{|b_i|^2}{\lambda_{i\bar{i}1}}$$

does not depend on a rescaling of (A, B) and is called the order of (A, B). If (A, B) is standard, then $o(B) = \sum_{i=1}^{k} |b_i|$. If (A, B) is normalized, then $o(B) = \sum_{i=1}^{k} |b_i|^2$.

Definition

A table algebra is called integral if all its degrees and structure constants are non-negative integers.

Examples: character algebra of a finite group

Let G be a finite group and $Ch(G)$ denote the algebra of all complex valued class functions on G with pointwise mutliplication. This algebra has a natural basis $Irr(G)$ consisting of irreducible characters of G. The pair $(Ch(G), Irr(G))$ satisfies the axioms of a table algebra. In this case $\bar{\chi}, \chi \in Irr(G)$ is a complex conjugate character and the degree function of χ is a usual degree of an irreducible character - $\chi(1)$. The algebra $(Ch(G), Irr(G))$ is a normalized integral table algebra (NITA, for short).

Examples: the center of a finite group algebra

Let G be a finite group and $Z(\mathbb{C}[G])$ denote the center of a group algebra. $Z(\mathbb{C}[G])$ is a subalgebra of $\mathbb{C}[G]$. Let $C_1 = \{1\}, C_2, ..., C_k$ be a complete set of conjugacy classes of G. Denote $b_i := \sum_{g \in C_i} g$, $Cla(G) := \{b_1, ..., b_k\}$. Then $Z((\mathbb{C}[G]), Cla(G))$ satisfies the axioms of a table algebra with $\overline{b_i} = \sum_{g \in C_i} g^{-1}$ and degree function $|b_i| = |C_i|$. The algebra $Z((\mathbb{C}[G]), Cla(G))$ is a standard integral table algebra (SITA, for short).

Table algebras classification results

Minimal degree

A minimal degree $m(B)$ of an ITA (A, B) is $\min\{|b_i| \mid i > 1\}$. ITAs containing a faithful element of degree 2 with $m(B) = 2$ were classified by Blau.

Homogeneous ITAs

HITAs of degrees $1, 2, 3$ were completely classified in a series of papers by Arad, Blau, Fisman, Miloslavsky and Muzychuk.

Standard ITAs

SITAs containing a faithful non-real element of minimal degree 3 and 4 were classified in a series of papers by Arad, Arisha, Blau, Fisman and Muzychuk.

Normalized integral table algebras

Let (A, B) be a NITA containing a faithful element b of minimal degree m. If $m = 1$, then (A, B) is exactly isomorphic to the character algebra of a cyclic group.If $m = 2$, then the classification of such algebras follows from Blau's result. In this talk we present the results obtained for $m = 3$ under additional assumption that b is non-real.

Normalized integral table algebras

Theorem (Arad, Chen)

Let (A, B) be a NITA of minimal degree 3 containing a faithful element b_3 of minimal degree 3. Then $b_3\overline{b_3} = 1 + b_8$ where $b_8 \in B$ is real of degree 8 and one of the following holds.

1. $(A, B) \cong_x ((Ch(G), Irr(G)), G \cong PSL(2, 7);$
2. $b_3^2 = b_4 + b_5$ where $b_4, b_5 \in B$;
3. $b_3^2 = c_3 + b_6$ where $c_3, b_6 \in B$, $c_3 \neq b_3, \bar{b}_3$;
4. $b_3^2 = \bar{b}_3 + b_6, b_6 \in B$ is non-real;

Theorem (Arad, Xu)

The second case cannot occur.

The third case

Theorem (Arad, Cohen, Arisha)

Assume that

$$b_3^2 = c_3 + b_6, c_3 \neq b_3, \bar{b}_3.$$

Then $(b_3 b_8, b_3 b_8) = 3, 4$. If $(b_3 b_8, b_3 b_8) = 3$ and c_3 is real, then there exists a unique NITA of dimension 22. If c_3 is not real, then there exists a unique NITA of dimension 32 satisfying these conditions. Both NITAs are not induced from character tables of finite groups.

Problem

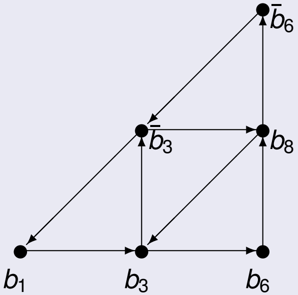
Classify the NITAs in the title with $(b_3 b_8, b_3 b_8) = 4$.

The fourth case

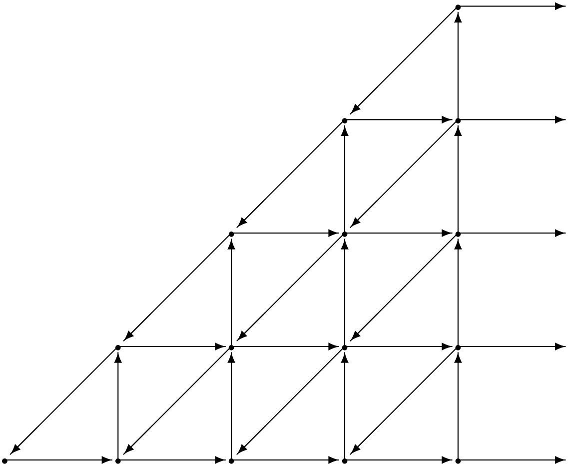
A representation graph

A representation graph of $b_i \in B$ is a weighted graph on B in which two vertices b_j and b_k are connected by an edge of weight λ_{ijk}.

A representation graph of b_3 at distance two



Graph C_n



Fourth case: $b_3^2 = \bar{b}_3 + b_6$

Definition

A NITA (A, B) in the title satisfies C_n -condition if the representation graph at distance n is isomorphic to C_n. We say that n is a stopping number for (A, B) if n is a maximal number for which (A, B) satisfies C_n-condition. In the case when (A, B) satisfies C_n-condition for each n, we say that its stopping number is ∞. In the latter case (A, B) is infinite dimensional algebra with $|B| = \aleph_0$,

Theorem (Arad, Cohen)

There exist only two algebras of fourth type with stopping number at most three, namely $(Ch(PSL(2,7)), Irr(PSL(2,7))$ and $(Ch(3 \cdot A_6), Irr(3 \cdot A_6)))$.

Fourth case: $b_3^2 = \bar{b}_3 + b_6$

Theorem (Arad, Cohen)

There exists no NITA of fourth type with stopping number at least 43.

Theorem (Arad, Cohen, Muzychuk)

There exists a unique infinite dimensional NITA of fourth type with stopping number ∞. This is the NITA of polynomial characters of $SL_3(\mathbb{C})$.

Open Problem

Classify all NITAs of fourth type with stopping number in the range $[4, 42]$.