On minimal normal subgroups of a TIN-group

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A subgroup H of a group G is *inert* if $|H : H \cap H^g|$ is finite for all $g \in G$ and a group G is *totally inert*, or briefly *TIN*, if every subgroup H of G is inert. Clearly finite groups and, more generally, *quasifinite* groups are totally inert. (Recall that a group is quasifinite if each of its proper subgroups is finite.) Moreover, it is easy to see that FC-groups are totally inert.
Note that the so-called Tarski monsters constructed by A. Yu. Olshanskii are quasifinite infinite simple groups and are therefore examples of infinite simple TIN-groups. On the other hand, we have:

Theorem (V.V. Belyaev, M. Kuzucuoğlu and E. Seçkin [BKS]).
There exist no infinite locally finite simple TIN-groups.

In [DET], we investigate the structure of minimal normal subgroups of totally inert groups and show that infinite locally graded simple groups cannot be totally inert.

Lemma A.
Let G be a locally residually finite group and M a nonabelian minimal normal subgroup of G that is not locally finite. Then M has infinitely many nonabelian, nonisomorphic finite simple sections.

Subgroups H and K of a group G are called *commensurable* if $|H : H \cap K|$ and $|K : K \cap H|$ are finite.

Theorem.
Let M be a minimal normal subgroup of a totally inert group G. Then either

- M is locally finite, or
- M is abelian and torsion-free, or
- M is a periodic group without involutions such that $C_M(x)$ is finite for all $1 \neq x \in M$ and in which every proper subgroup is residually finite.

Lemma B.
Let M be a minimal normal subgroup of a totally inert group G. Then the following assertions hold:

i. every proper subgroup of M is residually finite;
ii. if x and y are nontrivial elements of M, then $C_M(x)$ and $C_M(y)$ are commensurable;
iii. if $1 \neq x \in M$ then $C_M(x)$ is an FC-group;
iv. if M has a nontrivial periodic element whose centralizer in G is infinite, then M is locally finite.

Proof of the theorem.
Suppose that M is not locally finite. Then M is either periodic or torsion-free. For, if $x, y \in M$ are such that $1 \neq x$ is periodic and y has infinite order, then $C_M(x)$ is finite by (iv) of Lemma B whereas $C_M(y)$ is infinite; this is impossible by Lemma B (ii).

We consider the cases
(1) M is torsion-free and (2) M is periodic, separately.

Case (1): If $1 \neq z \in M$, then $C_M(z)$ is a (torsion-free) FC-group by (iii) of Lemma B and so it is abelian. Let x and y be nontrivial elements of M. Since Lemma B (ii) shows that $|C_M(x) : C_M(y) \cap C_M(y)| < \infty$, we have that $C_M(x) \cap C_M(y)$ is nontrivial. Let $1 \neq z \in C_M(x) \cap C_M(y)$. Then $x, y \in C_M(z)$, which is abelian, hence $xy = yx$ and M is abelian.

Case (2): By (iv) of Lemma B, $C_M(x)$ is finite for all $1 \neq x \in M$ and a result of Šunkov guarantees that M has no involutions. Finally, all proper subgroups of M are residually finite by Lemma B (i). □

On appealing to the theorem above (with $G = M$) and using Lemma A, we deduce:

Corollary A.
If G is an infinite totally inert simple group, then G is finitely generated and periodic. In particular, it is not locally graded.

Corollary B.
Let G be a simple group of finite exponent. Then G is totally inert if and only if it is quasifinite.

It remains an open question whether simple TIN-groups are necessarily quasifinite. However, it follows from a result in [BL] that a simple TIN-group G is quasifinite if the set $\{|H : H \cap H^g| : g \in G\}$ is finite for each subgroup H of G.

References:

[BKS] V.V. Belyaev, M. Kuzucuoğlu and E. Seçkin, *Totally inert groups*, Rend. Sem. Mat. Univ. Padova **102** (1999), 151–156.
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