Automorphisms of Groups of Odd Order

OLIVEIRA, Karise Gonçalves[1]

Introduction

Let G be a finite group of odd order . It is well-known that if a is an involutory automorphism of G, then the properties of $C_G(a)$ have strong impact over the structure of G.

It was shown by Shumyatsky [2] that if G is a finite group of derived length k and if G admits a fixed-point-free action of the elementary group of order 2^n, then G has a normal series of length n all of whose quotients are nilpotent of class bounded in terms of k and n only. Recently a shorter proof of this result was obtained in Shumyatsky and Sica [3].

We consider the situation when the elementary group A of order 2^n acts on a group G of odd order in such way that $C_G(A)$ has exponent m. Our main result is the following theorem

Theorem 1 *Let G be a finite group of odd order and of derived length k. Let A be the elementary group of order 2^n acting on G in such way that $C_G(A)$ has exponent m. Then G has a normal series*

$$G = G_1 \geq T_1 \geq G_2 \geq T_2 \geq ... \geq G_n \geq T_n = 1$$

such that the quotients G_i/T_i are nilpotent of $\{k, m, n\}$-bounded class and the quotients T_i/G_{i+1} have $\{k, m, n\}$-bounded exponent.

The proof of the above theorem is based on the same techniques as [2] and [3].

Preliminaries

The following three lemmas are well-known (see for example [4, 6.2.2,6.2.4,10.4.1]).

Lemma 2 *Let G be a group and N a normal subgroup of G such that $N = n$ and G/N has exponent d. Then the exponent of G is less or iqual than $n \cdot d$.*

Lemma 3 *Let G be a finite group admitting a coprime finite group of automorphisms A. Then we have*

a) $C_{G/N}(A) = C_G(A)N/N$ *for any A-invariant normal subgroup N of G;*

b) $G = C_G(A)[G, A]$;

c) $[G, A] = [G, A, A]$;

d) If G is abelian then $G = C_G(A) \oplus [G, A]$.

Lemma 4 *Let G be a finite group of odd order admitting an automorphism a of order 2 such that $G = [G, a]$. Suppose that N be an a-invariant normal subgroup of G such that $C_N(a) = 1$. Then $N \leq Z(G)$.*

Some Results

We are now ready to formulate the key theorem. At first we formulate the Theorem 6 under the hypothesis G being metabelian. It will be done in the next lemma.

Lemma 5 *Let G be a metabelian finite group of odd order admitting an automorphism a of order 2 such that $C_G(a)$ has exponent m and $G = [G, a]$. Then G' has exponent m.*

Proof. By the hypothesis $G = [G, a]$ it follows that $C_G(a) \subseteq G'$. Let N be the normal closure of $C_G(a)$ in G. Since G' is abelian we conclude that N has exponent m. By the other hand G/N is abelian because a acts on G/N fixed-point-free. Then $G' = N$. □

Theorem 6 *Let G be a finite group of odd order and of derived length k, admitting an automorphism a of order 2 such that $C_G(a)$ has exponent m and $G = [G, a]$. Then G' has $\{m, k\}$-bounded exponent.*

Proof. We notice that for $k \leq 2$ the result follows by the Proposition 5. So we assume that $k \geq 3$ and use induction on k. By induction we conclude that $G/G^{(k-1)}$ has $\{m, k\}$-bounded exponent.
So it is suficient to show that $G^{(k-1)}$ has $\{m, k\}$-bounded exponent. Consider $G/\left\langle C_{G^{(k-1)}}(a)^G \right\rangle$. Without loss of generality we suppose that $C_{G^{(k-1)}}(a) = 1$, so $G^{(k-1)} \leq Z(G)$. Then $G^{(k-2)}$ is nilpotent with class 2. Since $G^{(k-2)} \leq G'$ it follows by induction hypothesis that $G^{(k-2)}/G^{(k-1)}$ has $\{m, k\}$-bounded exponent. By [5, Theorem 2.5.2] it follows that $G^{(k-2)}$ has $\{m, k\}$-bounded exponent. Then also has $G^{(k-1)}$. □

Theorem 7 *Let c, d, q be positive integers. Suppose that G is a soluble group with derived length d. Assume that for any i the metabelian quotient $G^{(i)}/G^{(i+2)}$ is an extention of a group of finite exponent q by a nilpotent group of class c. Then there exist $\{c, d, q\}$-bounded numbers f and g such that G is an extension of a group of finite exponent g by a nilpotent group of class f.*

A well-known theorem of Hall [6] says that: if G is a soluble group of derived length k, and all metabelian sections of G are nilpotents of class at most c, then G is nilpotent with $\{k, c\}$-bounded nilpotency class. We use the related result above in the proof of the next proposition.

Proposition 8 *Let G be a finite group of odd order and with derived length k, admitting an elementary abelian group of automorphisms A of order 2^n such that $C_G(A)$ has exponent m. Then $N = \bigcap_{a \in A^\#} [G, a]$ is an extension of a group of $\{k, m, n\}$-bounded exponent by a nilpotent group of $\{k, m, n\}$-bounded class.*

Sketch of Proof of the Theorem 1

We use induction on n. For $n = 1$ the Proposition 6 guarantes that G' has $\{m, k\}$-bounded exponent, and the result follows.

Suppose that $n \geq 2$ and the result is true for any group admitting an elementary abelian group of automorphisms of order less or iqual 2^{n-1}. For any normal A-invariant subgroup T of G the group A induces a group of automorphisms of G/T. In particular A induces a group of automorphisms of $G/[G, a]$ for each $a \in A^\#$.

Let B be the elementary group of order 2^{n-1}. So, for each element $a \in A^\#$ exists the corresponding action of B on $G/[G, a]$. Let $K = \prod_{a \in A^\#} G/[G, a]$. We have an action of B on K. By the induction hypothesis K has a series of length $2(n-1)$ with the required conditions. Let $N = \bigcap_{a \in A^\#} [G, a]$.

Proposition 8 tell us that N is an extension of a group of $\{k, m, n\}$-bounded exponent by a nilpotent group of $\{k, m, n\}$-bounded class. Since G/N embeds in K the result follows.

References

[1] SHUMYATSKY, P., *On extensions of groups of finite exponent*, Glasgow Math J. **45** (2003), 535-538.

[2] SHUMYATSKY, P., *Groups with regular elementary 2-groups of automorphisms*, Algebra and Logic, **27** no. 6, (1988), 715-730.

[3] SHUMYATSKY, P. and Sica, C., *On groups admitting a fixed- point-free elementary 2-group of automorphisms*, pre-print.

[4] GORENSTEIN, D., *Finite Groups*, Harper and Row, New York, Evanston, London, 1968.

[5] KHUKHRO, E., *Nilpotent groups and their automorphisms*, Berlin, New York, De Gruyter expositions in mathematics; **8**, 1993.

[6] HALL, P., *Some Sufficient Conditions for a Group to be Nilpotent*, Illinois J. Math., **2** (1958), 787-801.

1: Federal Institute of Education, Science and Technology of Goias, Brazil; email: karisemat@gmail.com. This work is part of PhD thesis written under supervision of Prof. Shumyatsky at University of Brasilia.