Examples of θ-Pairs

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The Definition of a θ-Pair

In [1] Xianhua Li and Shiheng Li, generalizing a concept from [2], gave the definition of a θ-pair (C, D) for a proper subgroup H of a finite group G:

(1) C is a subgroup of H and $G = \langle C, H\rangle$.

(2) The normal closure $D := (C\cap H)_G$ of $C\cap H$ in G is a subgroup of C. Whenever $C_1/D < C/D$ then $\langle C_1, H\rangle < G$.

A Simple Example

Let G be a finite non-abelian simple group and P_1 any maximal subgroup of a non-cyclic Sylow subgroup P. Then $(C, D) := (G, (P_1)_G) = (G, G)$ is a θ-pair for P_1.

A Solvable Example

Let $V = GF(3) \oplus GF(3)$ be the vector space of dimension 2 over the 3-element field $GF(3)$ and $x := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ Note that $\langle x\rangle \cong C_4$, the cyclic group of order 4 and that V does not contain an x-invariant proper subspace.
The only non-cyclic Sylow subgroup of G is V. Any maximal subgroup P_1 of V is a 1-dimensional subspace and hence $D := (P_1)_G$ coincides with V. Setting $C := G$ it turns out that (C, D) is a θ-pair for P_1.
Every 2-Sylow subgroup of G is conjugate to $\langle x\rangle$. The only maximal subgroup P_1 of $P := \langle x\rangle$ coincides with $\langle x^2\rangle$ and $D := (P_1)_G = V\langle x^2\rangle$.

Discussion

•1 The first example is not supersolvable since it is a finite non-abelian simple group.
•• The second example is not supersolvable since V does not possess a 1-dimensional x-invariant subspace.
••• In both examples for every θ-pair (C, D) the factor group C/D is supersolvable.

Conclusion

In [2], N.P. Mukherjee and P. Bhattacharya conclude from $|C/D|$ a prime for every maximal θ-pair (C, D) of a maximal subgroup M that G is supersolvable. However, in *their* definition of a θ-pair they stipulate the stronger condition $D < C$.
In [1] they authors show that G is supersolvable if for every maximal subgroup of every non-cyclic Sylow subgroup there is a θ-pair (C, D) with C/D supersolvable. It is this result that confuses me in light of the above examples.

References

[1] X. Li and S. Li, *Theta pairs and the structure of finite groups*, Sib.Math.J. 45(2004)
[2] N.P. Mukherjee and P. Bhattacharya, *On theta pairs for a maximal subgroup*, Proc.Am.Soc. 109(1990)