Nilpotent injectors and strongly closed subgroups

A.L.Gilotti and L. Serena

1.- Introduction.

Let G be a finite group. A set \mathcal{F} of subgroups of G is said to be a Fitting set of G if the following conditions are satisfied:

- If $A \in \mathcal{F}$ and B is a subnormal subgroup of A then $B \in \mathcal{F}$
- If A and B are in \mathcal{F} and A and B are normal in AB, then $AB \in \mathcal{F}$.
- if $A \in \mathcal{F}$ and $g \in G$ then $A^g \in \mathcal{F}$.

Some examples of Fitting sets in a finite group G are:

 $\mathcal{N} = \{ \text{ all nilpotent subgroups of } G \}.$

 $S_{pi} = \{ all \ \pi\text{-subgroups of } G \} \ (\pi \text{ is a set of primes which divide the order of } G).$

 $s_n(N) = \{ all subnormal subgroups of N where N is a normal subgroup of G \}.$

A subgroup H of G is an \mathcal{F} -injector of G (or H is an injector of G with respect to \mathcal{F}) if

 $H \cap X$ is \mathcal{F} - maximal in X for every subnormal subgroup X of G.

A subgroup H of G is said to be an injector of G if it is an \mathcal{F} - injector for some Fitting set \mathcal{F} of G.

The theory of Fitting sets and of their injectors, which generalizes that of Fitting class and their injectors, plays an important role in the context of finite soluble groups. If G is soluble and \mathcal{F} is a Fitting set of G, then G admits \mathcal{F} -injectors and any two \mathcal{F} -injectors of G are conjugate (see [4]). In particular if G is a finite soluble group then the \mathcal{N} - injectors of G are precisely the maximal nilpotent subgroups which contain the Fitting subgroup F(G) and the S_{π} -injectors of G are precisely the Hall π - subgroups of G. Doerk and Hawkes in [4][p.549] gave a very useful characterization of injectors of a finite soluble group G.

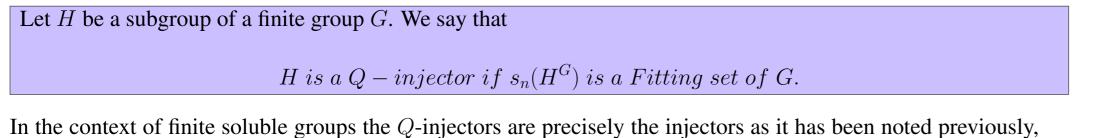
Let

 $s_n(H^G) = \{S \le G : S \text{ is subnormal in } H^g \text{ for some } g \in G\},\$

then a subgroup H of G is an injector of G if and only if

 $s_n(H^G)$ is a Fitting set of G

Doerk and Hawkes [4] raised the question to describe the injectors of finite soluble groups without recourse to the concept of a Fitting set. This was settled by R.Dark and A. Feldman [1] and successively by R.Dark, A.Feldman and M.D.Pérez-Ramos [2].



but the two concepts are not equivalent if G is an arbitrary finite group. However the nilpotent Q-injectors are

Anna Luisa Gilotti, Università degli Studi di Bologna, e-mail: gilotti@dm.unibo.it

We give a characterization of nilpotent Q-injectors in finite groups, without reference to Fitting sets. For this, recall that:









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injectors.

If G is a finite soluble groups and H is a nilpotent subgroup of G, then H is an injector of G if and only if it is normally embedded, that is any Sylow *p*-subgroup of H is a Sylow *p*-subgroup of its normal closure in G (*p* is a prime which divides |H|).

2.- Results.

If G is a finite group and P is a Sylow p-subgroup of G, then a subset A of P is said to be strongly closed in Pwith respect to G if for any $g \in G$ we have $A^g \cap P \subseteq A$.

Observe that if A is strongly closed in P (with respect to G) then A is strongly closed in any Sylow p-subgroup containing it. For this reason we simply say that A is strongly closed.

Then we have the following

Theorem Let G be a finite group and $H \leq G$ with H nilpotent. Then H is a Q-injector of G if and only if H_p is strongly closed for all $p \in \pi(G)$ (H_p indicates a Sylow *p*-subgroup of H)

Using this characterization we have that

A nilpotent maximal subgroup of a finite group G is a Q-injector.

Further we obtain the following

Theorem Let H be a nilpotent maximal subgroup of a finite group G. Suppose that H possesses a central chain of Q-injectors. Then G is soluble.

As an application we have that if $H = P_1 \times P_2 \cdots \times P_r$ is a nilpotent maximal subgroup of a finite group G $(P_i \in Syl_{p_i}(H))$ and

$$1 < \Omega_1(P_i) \le \Omega_2(P_i) \le \dots \le \Omega_t(P_i) = P_i$$

is a central chain of P_i , $(\Omega_i(P_i) = \langle x : x^{p_i^j} = 1 \rangle)$, then

$$1 < H_1 \le H_2 \le \dots \le H_t = H,$$

where $H_i = \Omega_i(P_1) \times \Omega_i(P_2) \cdots \times \Omega_i(P_r)$, is a central chain of H consisting of Q-injectors of G. In particular if H is a nilpotent maximal subgroup of a finite group G and the Sylow subgroups of H are either abelian, quaternion or uniformly powerful then G is soluble.

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Luigi Serena, Università degli Studi di Firenze, e-mail: luigi.serena@unifi.it