Nilpotent injectors and strongly closed subgroups

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1.- Introduction.

Let G be a finite group. A set $\mathcal{F}$ of subgroups of G is said to be a Fitting set of G if the following conditions are satisfied:

- If $A \in \mathcal{F}$ and B is a subnormal subgroup of A then $B \in \mathcal{F}$
- If A and B are in $\mathcal{F}$ and A and B are normal in AB, then $AB \in \mathcal{F}$.
- if $A \in \mathcal{F}$ and $g \in G$ then $A^g \in \mathcal{F}$.

Some examples of Fitting sets in a finite group G are:
$\mathcal{N}$= { all nilpotent subgroups of G}.
$\mathcal{S}_{pi}$= {all π-subgroups of G} (π is a set of primes which divide the order of G).
$s_n(N)$={ all subnormal subgroups of N where N is a normal subgroup of G}.

A subgroup H of G is an $\mathcal{F}$-injector of G (or H is an injector of G with respect to $\mathcal{F}$) if

$$H \cap X \text{ is } \mathcal{F}\text{- maximal in X for every subnormal subgroup } X \text{ of } G.$$

A subgroup H of G is said to be an injector of G if it is an $\mathcal{F}$- injector for some Fitting set $\mathcal{F}$ of G.

The theory of Fitting sets and of their injectors, which generalizes that of Fitting class and their injectors, plays an important role in the context of finite soluble groups. If G is soluble and $\mathcal{F}$ is a Fitting set of G, then G admits $\mathcal{F}$-injectors and any two $\mathcal{F}$-injectors of G are conjugate (see [4]). In particular if G is a finite soluble group then the $\mathcal{N}$- injectors of G are precisely the maximal nilpotent subgroups which contain the Fitting subgroup $F(G)$ and the $\mathcal{S}_\pi$-injectors of G are precisely the Hall π- subgroups of G. Doerk and Hawkes in [4][p.549] gave a very useful characterization of injectors of a finite soluble group G.

Let

$$s_n(H^G) = \{S \leq G : S \text{ is subnormal in } H^g \text{ for some } g \in G\},$$

then a subgroup H of G is an injector of G if and only if

$$s_n(H^G) \, is\ a\ Fitting\ set\ of\ G$$

Doerk and Hawkes [4] raised the question to describe the injectors of finite soluble groups without recourse to the concept of a Fitting set. This was settled by R.Dark and A. Feldman [1] and successively by R.Dark, A.Feldman and M.D.Pérez-Ramos [2] .

Let H be a subgroup of a finite group G. We say that

$$H\ is\ a\ Q-injector\ if\ s_n(H^G)\ is\ a\ Fitting\ set\ of\ G.$$

In the context of finite soluble groups the Q-injectors are precisely the injectors as it has been noted previously, but the two concepts are not equivalent if G is an arbitrary finite group. However the nilpotent Q-injectors are injectors.

If G is a finite soluble groups and H is a nilpotent subgroup of G, then H is an injector of G if and only if it is normally embedded, that is any Sylow p-subgroup of H is a Sylow p-subgroup of its normal closure in G (p is a prime which divides $|H|$).

2.- Results.

We give a characterization of nilpotent Q-injectors in finite groups, without reference to Fitting sets. For this, recall that:

If G is a finite group and P is a Sylow p-subgroup of G, then a subset A of P is said to be strongly closed in P with respect to G if for any $g \in G$ we have $A^g \cap P \subseteq A$.

Observe that if A is strongly closed in P (with respect to G) then A is strongly closed in any Sylow p-subgroup containing it. For this reason we simply say that A is strongly closed.

Then we have the following

Theorem *Let G be a finite group and $H \leq G$ with H nilpotent. Then H is a Q-injector of G if and only if H_p is strongly closed for all $p \in \pi(G)$* (H_p indicates a Sylow p-subgroup of H)

Using this characterization we have that

A nilpotent maximal subgroup of a finite group G is a Q-injector.

Further we obtain the following

Theorem *Let H be a nilpotent maximal subgroup of a finite group G. Suppose that H possesses a central chain of Q-injectors. Then G is soluble.*

As an application we have that if $H = P_1 \times P_2 \cdots \times P_r$is a nilpotent maximal subgroup of a finite group G ($P_i \in Syl_{p_i}(H)$) and

$$1 < \Omega_1(P_i) \leq \Omega_2(P_i) \leq \cdots \leq \Omega_t(P_i) = P_i$$

is a central chain of P_i, ($\Omega_j(P_i) = \langle x : x^{p_i^j} = 1 \rangle$), then

$$1 < H_1 \leq H_2 \leq \cdots \leq H_t = H,$$

where $H_i = \Omega_i(P_1) \times \Omega_i(P_2) \cdots \times \Omega_i(P_r)$, is a central chain of H consisting of Q-injectors of G. In particular if H is a nilpotent maximal subgroup of a finite group G and the Sylow subgroups of H are either abelian, quaternion or uniformly powerful then G is soluble.

References

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