# The multiplication groups of 2-dimensional topological loops

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**Abstract**

If the multiplication group $Mult(L)$ of a connected simply connected 2-dimensional topological loop $L$ is a Lie group, then $Mult(L)$ is an elementary filiform Lie group $\mathcal{F}$ of dimension $n+2$, $n \geq 2$, and any such group is the multiplication group of a connected simply connected 2-dimensional topological loop $L$. Moreover, if the group topologically generated by the left translations of $L$ has dimension 3, then $L$ is uniquely determined by a real polynomial of degree $n$.

## Introduction

The multiplication group $Mult(L)$ and the inner mapping group $Inn(L)$ of a loop $L$ introduced in [1], [2] are important tools for research since they strongly reflect the structure of $L$. In particular, there is a strong correspondence between the normal subloops of $L$ and certain normal subgroups of $Mult(L)$. Hence, it is an interesting question which groups can be represented as multiplication groups of loops ([7], [8], [9]). A purely group theoretic characterization of multiplication groups is given in [7].

Topological and differentiable loops such that the groups $G$ topologically generated by the left translations are Lie groups have been studied in [5]. There the topological loops $L$ are treated as continuous sharply transitive sections $\sigma : G/H \to G$, where $H$ is the stabilizer of the identity element of $L$ in $G$. In [5] and [3] it is proved that essentially up to two exceptions any connected 3-dimensional Lie group occurs as the group topologically generated by the left translations of a connected 2-dimensional topological loop. These exceptions are either locally isomorphic to the connected component of the group of motions or isomorphic to the connected component of the group of dilatations of the euclidean plane. In contrast to this, if the group $Mult(L)$ topologically generated by all left and right translations of a connected 2-dimensional topological loop $L$ is a Lie group, then the isomorphism types of $Mult(L)$ and of $L$ are strongly restricted. This is shown by our theorems, in which Lie groups with filiform Lie algebras (cf. [4], pp. 626-663) play a fundamental role.

The elementary filiform Lie group $\mathcal{F}_{n+2}$ is the simply connected Lie group of dimension $n+2 \geq 3$ whose Lie algebra is elementary filiform, i.e. it has a basis $\{e_1, \cdots, e_{n+2}\}$ such that $[e_1, e_i] = (n+2-i)e_{i+1}$ for $2 \leq i \leq n+1$ and all other Lie brackets are zero. With this notion we can formulate our theorems as follows:

**Theorem 1.** *Let $L$ be a connected simply connected 2-dimensional topological loop which is not a group. The group $Mult(L)$ topologically generated by all left and right translations of $L$ is a Lie group if and only if $Mult(L)$ is an elementary filiform Lie group $\mathcal{F}_{n+2}$ with $n \geq 2$. Moreover, the group $G$ topologically generated by the left translations of $L$ is an elementary filiform Lie group $\mathcal{F}_{m+2}$, where $1 \leq m \leq n$, and the inner mapping group $Inn(L)$ corresponds to the abelian subalgebra $\langle e_2, e_3, \cdots, e_{n+1}\rangle$.*

The loop $L$ of Theorem 1 is a central extension of the group $\mathbb{R}$ by the group $\mathbb{R}$ (cf. Theorem 28.1 in [5], p. 338). Hence it is a centrally nilpotent loop of class 2 and can be represented in $\mathbb{R}^2$. If $L$ is not simply connected but satisfies all other conditions of Theorem 1, then $L$ is homeomorphic to the cylinder $\mathbb{R} \times \mathbb{R}/\mathbb{Z}$.

**Theorem 2.** *Let $G$ be the elementary filiform Lie group $\mathcal{F}_{n+2}$ with $n \geq 1$. Then $G$ is isomorphic to the group topologically generated by the left translations of a connected simply connected 2-dimensional topological loop $L = (\mathbb{R}^2, *)$ with the multiplication*

$$(u_1, z_1)*(u_2, z_2) = (u_1+u_2, z_1+z_2-u_2v_1(u_1)+u_2^2v_2(u_1)+\cdots+(-1)^n u_2^n v_n(u_1)), \quad (1)$$

*where $v_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \cdots, n$, are continuous functions with $v_i(0) = 0$ such that the function $v_n$ is non-linear.*

*For $n > 1$ the group $G$ coincides with the group $Mult(L)$ topologically generated by all left and right translations of $L$ if and only if there are continuous functions $s_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \cdots, n$, such that for all $x, u \in \mathbb{R}$ the equation*

$$-x(s_1(u)+v_1(u)) + x^2(s_2(u)+v_2(u)) + \cdots + (-1)^n x^n (s_n(u)+v_n(u))$$

$$= -uv_1(x) + u^2v_2(x) + \cdots + (-1)^n u^n v_n(x)$$

*holds.*

**Theorem 3.** *If $L$ is a 2-dimensional connected simply connected topological loop having the elementary filiform Lie group $\mathcal{F}_3$ as the group topologically generated by the left translations of $L$, then the multiplication of $L$ is given by*

$$(u_1, z_1) * (u_2, z_2) = (u_1 + u_2, z_1 + z_2 - u_2 v_1(u_1)), \qquad (2)$$

*where $v_1 : \mathbb{R} \to \mathbb{R}$ is a non-linear continuous function with $v_1(0) = 0$.*

*The group $Mult(L)$ topologically generated by all left and right translations of $L$ is isomorphic to the elementary filiform Lie group $\mathcal{F}_{n+2}$ for $n \geq 2$ if and only if the continuous function $v_1 : \mathbb{R} \to \mathbb{R}$ is a polynomial of degree $n$.*

## Basic facts in loop theory

A binary system $(L, \cdot)$ is called a loop if there exists an element $e \in L$ such that $x = e \cdot x = x \cdot e$ holds for all $x \in L$ and the equations $a \cdot y = b$ and $x \cdot a = b$ have precisely one solution, which we denote by $y = a \backslash b$ and $x = b/a$. A loop $L$ is proper if it is not associative.
The left and right translations $\lambda_a = y \mapsto a \cdot y : L \to L$ and $\rho_a : y \mapsto y \cdot a : L \to L$, $a \in L$, are permutations of $L$. The permutation group $Mult(L) = \langle \lambda_a, \rho_a;\ a \in L \rangle$ is called the multiplication group of $L$. The stabilizer of the identity element $e \in L$ in $Mult(L)$ is denoted by $Inn(L)$, and $Inn(L)$ is called the inner mapping group of $L$.

Let $K$ be a group, let $S \leq K$, and let $A$ and $B$ be two left transversals to $S$ in $K$ (i.e. two systems of representatives for the left cosets of the subgroup $S$ in $K$). We say that the two left transversals $A$ and $B$ are $S$-connected if $a^{-1}b^{-1}ab \in S$ for every $a \in A$ and $b \in B$. By $C_K(S)$ we denote the core of $S$ in $K$ (the largest normal subgroup of $K$ contained in $S$). If $L$ is a loop, then $\Lambda(L) = \{\lambda_a;\ a \in L\}$ and $R(L) = \{\rho_a;\ a \in L\}$ are $Inn(L)$-connected transversals in the group $Mult(L)$, and the core of $Inn(L)$ in $Mult(L)$ is trivial. The connection between multiplication groups of loops and transversals is given in [7] by Theorem 4.1. This theorem yields the following lemma which is important tool to prove Theorems 2 and 3.

**Lemma 4.** *Let $L$ be a loop and $\Lambda(L)$ be the set of left translations of $L$. Let $K$ be a group containing $\Lambda(L)$ and $S$ be a subgroup of $K$ with $C_K(S) = 1$ such that $\Lambda(L)$ is a left transversal to $S$ in $K$. The group $K$ is isomorphic to the multiplication group $Mult(L)$ of $L$ if and only if there is a left transversal $T$ to $S$ in $K$ such that $\Lambda(L)$ and $T$ are $S$-connected and $K = \langle \Lambda(L), T \rangle$. In this case $S$ is isomorphic to the inner mapping group $Inn(L)$ of $L$.*

The kernel of a homomorphism $\alpha : (L, \circ) \to (L', *)$ of a loop $L$ into a loop $L'$ is a normal subloop $N$ of $L$. To the proof of Theorem 1 we need the following lemma and the classification of Lie transformation groups in [6].

**Lemma 5.** *Let $L$ be a loop with multiplication group $Mult(L)$ and identity element $e$.*

*(i) Let $\alpha$ be a homomorphism of the loop $L$ onto the loop $\alpha(L)$ with kernel $N$. Then $\alpha$ induces a homomorphism of the group $Mult(L)$ onto the group $Mult(\alpha(L))$.*

*Denote by $M(N)$ the set $\{m \in Mult(L);\ xN = m(x)N$ for all $x \in L\}$. Then $M(N)$ is a normal subgroup of $Mult(L)$, and the multiplication group of the factor loop $L/N$ is isomorphic to $Mult(L)/M(N)$.*

*(ii) For every normal subgroup $\mathcal{N}$ of $Mult(L)$ the orbit $\mathcal{N}(e)$ is a normal subloop of $L$. Moreover, $\mathcal{N} \leq M(\mathcal{N}(e))$.*

The theory of topological loops $L$ is the theory of the continuous binary operations $(x, y) \mapsto x \cdot y$, $(x, y) \mapsto x/y$, $(x, y) \mapsto x\backslash y$ on the topological space $L$. If $L$ is a topological loop, then the left translations $\lambda_a$ as well as the right translations $\rho_a$, $a \in L$, are homeomorphisms of $L$.

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