Centralizers of Finite Subgroups in Simple Locally Finite Groups

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A group G is called locally finite if every finitely generated subgroup of G is finite. In this work, we present some results on centralizers in simple locally finite groups proved in [3].

First, let's consider some examples of simple locally finite groups:

Example 1. Let Ω be an infinite set. The alternating group on the set Ω, which is denoted by $Alt(\Omega)$, is a simple locally finite group with cardinality $|\Omega|$.

Example 2. A field is called locally finite if every finitely generated subfield is finite. Let $\mathbb{F}$ be an infinite locally finite field. The group $PSL_n(\mathbb{F})$ is a simple locally finite group.

Example 3. Let $\mathbb{F}$ be a finite field. Observe that the map $\phi_n : SL_n(\mathbb{F}) \longrightarrow SL_{n+1}(\mathbb{F})$ which sends each $A \in SL_n(\mathbb{F})$ to $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in SL_{n+1}(\mathbb{F})$ embeds $SL_n(\mathbb{F})$ into $SL_{n+1}(\mathbb{F})$.
The direct limit of the directed system $(SL_n(\mathbb{F}), \phi_n)$, is a simple locally finite group denoted by $\mathbf{SL^0}(\mathbb{F})$ and called the **Stable Special Linear Group**.

Locally finite groups are infinite groups with a finiteness condition, hence it is possible to use some information about finite groups to understand the structure of locally finite groups. Since finite simple groups are classified, it is natural to ask if it is possible to classify all infinite simple locally finite groups too. The experts think that we are far from an answer to this question. By [9, Corollary 6.12], there exist $2^{\aleph_0}$ non-isomorphic countable simple locally finite groups which can be obtained as direct limits of finite alternating groups. However, it may be possible to obtain information about some "nice" families of simple locally finite groups, and try to generalize the scope of this.

A group G is called **linear** if it has a faithful representation into $GL_n(k)$ for some natural number n and for some field k. Linear simple locally finite groups were classified independently by Belyaev, Borovik, Hartley-Shute and Thomas (see [1, 2, 8, 13]). They proved that a linear simple locally finite group is a Chevalley or a twisted Chevalley group over a locally finite field.

The following result of Turau enables us to see the structure of locally finite simple groups of Lie type.

Theorem 4. *[7, Lemma 4.3] Let G be a Chevalley group (or a twisted Chevalley group) over an infinite locally finite field k of characteristic p and let $\overline{G}$ be the simple algebraic group over the algebraic closure $\overline{k}$ of k constructed from the same Lie algebra as G. Then there are a Frobenius map σ and a sequence $n_1, n_2, \ldots$ of positive integers such that n_i divides n_{i+1} and $G = \cup_{i=1}^{\infty} G_i$ where $G_i = O^{p'}(C_{\overline{G}}(\sigma^{n_i}))$.*

Observe that by Theorem 4, we can write a linear simple locally finite group as a union of finite simple groups of the same Lie type. Also, the simple locally finite groups in Example 1 and 3 are examples of non-linear simple locally finite groups which can be written as a union of finite simple groups.

Kegel-Wehrfritz asked the following question:

Question 5. *Does every simple locally finite group has a local system consisting of finite simple groups?*

Serezhkin and Zalesskii answered this question negatively in [14]. They proved the following result:

Theorem 6. *[6, Proposition 1.7] If k is a finite field of odd order then the Stable Symplectic Group is an infinite simple locally finite group which can not be written as a union of finite simple groups.*

So, there are simple locally finite groups which can not be written as a union of finite simple groups. But still, there is a useful concept that connects the theory of finite simple groups and theory of infinite simple locally finite groups, namely **Kegel covers**.

The following result is very useful to understand the structure of infinite simple groups:

Theorem 7. *[9, Theorem 4.4] An infinite group G is simple iff it has a local system consisting of countably infinite simple subgroups of G.*

By Theorem 7, an infinite group is simple iff it has a local system of countably infinite simple groups. Hence every finite subset of an infinite simple group is contained in a countably infinite simple group. Hence, to answer Question 18, it will enough to consider the centralizers of finite subgroups in countable simple locally finite groups. So, for simplicity, we can define Kegel covers for countably infinite locally finite groups:

Definition 8. *[6, Definition 2.2] Let G be a countable locally finite group. A set $\{(G_i, N_i) \mid i \in \mathbb{N}\}$ consisting of pairs of finite subgroups of G satisfying $N_i \trianglelefteq G_i$, is called a Kegel cover of G if*

$$G = \bigcup_{i \in I} G_i$$

the factors G_i/N_i are finite simple groups and $G_i \cap N_{i+1} = 1$.

Theorem 9. *[9, Kegel-Wehrfritz, Lemma 4.5] Every countable simple locally finite groups has a Kegel cover $\mathcal{K} = \{(G_i, N_i) \mid i \in \mathbb{N}\}$.*

Centralizers in Simple Locally Finite Groups

Kegel and Wehrfritz asked the following question in [9]:

Question 10. *If G is an infinite simple locally finite group, is the centralizer of an element necessarily infinite?*

Hartley and Kuzucuoğlu answered Question 10 positively, namely they proved in [7, Theorem A2] that in an infinite locally finite simple group, the centralizer of every element is infinite. Hartley proved in [4, Corollary A1] that if G is a locally finite group containing an element with finite centralizer, then G contains a locally solvable normal subgroup of finite index. Then, it is natural to ask the same question for the fixed points of automorphisms in simple locally finite groups and for centralizers of finite subgroups. Hartley proved the following result on fixed points of automorphisms in linear simple locally finite groups:

Theorem 11. ***(Hartley, [4, Theorem C])*** *Let G be an infinite simple group of Lie type over a locally finite field K of characteristic p and let α be an automorphism of finite order n of G. Suppose that p does not divide n. Then there exists infinitely many primes q such that α fixes an element of order q.*

Hartley asked the following question in [5]:

Question 12. *Let G be a non-linear simple locally finite group and F be a finite subgroup of G. Is $C_G(F)$ necessarily infinite?*

One may think about the linear version of this question, which has a negative answer. Indeed, a linear simple locally finite group is a subset of the fixed points of powers of a Frobenius map in a simple linear algebraic group. Here, we can first find the centralizers in the linear algebraic group, and then intersect with the fixed points of the Frobenius maps. A linear algebraic group is an affine variety and the centralizers of elements are closed subsets. By Hilbert Basis Theorem, the closed subsets of an algebraic variety satisfy descending chain condition. So, one can always construct a finite subgroup of a linear simple locally finite group, with trivial centralizer. Hence, in linear case we restrict our attention to abelian subgroups consisting of semisimple elements.

Question 13. *Let G be a linear simple locally finite group. Determine all the finite abelian subgroups A consisting of semisimple elements such that $C_G(A)$ contains an infinite abelian subgroup isomorphic to the direct product of cyclic groups of order p_i for infinitely many primes p_i.*

Definition 14. *Let $\overline{G}$ be a simple linear algebraic group. A finite abelian subgroup A consisting of semisimple elements of $\overline{G}$ is called a* d***-abelian*** *subgroup if it satisfies one of the following:*

1. *The root system associated with $\overline{G}$ has type A_l and Hall-π-subgroup of A is cyclic where π is the set of primes dividing $l+1$*
2. *The root system associated with $\overline{G}$ has type B_l, C_l, D_l or G_2 and the Sylow 2-subgroup of A is cyclic.*
3. *The root system associated with $\overline{G}$ has type E_6, E_7 or F_4 and the Hall-$\{2,3\}$-subgroup of A is cyclic.*
4. *The root system associated with $\overline{G}$ has type E_8 and the Hall-$\{2,3,5\}$-subgroup of A is cyclic.*

We extend Hartley's result Theorem 11 to centralizers of d-abelian subgroups of linear simple locally finite groups by proving the following theorem:

Theorem 15. *[3] Let G be a locally finite simple group of Lie type defined over an infinite locally finite field of characteristic p. Let A be a d-abelian subgroup of G. Then $C_G(A)$ contains infinitely many commuting elements of distinct prime orders.*

Now, we present an example of a non-d-abelian subgroup A such that $C_{\overline{G}}(A) = A$.

Example 16. Let $\overline{G}$ be the adjoint group $A_1(K) = PSL_2(K)$ defined over an algebraically closed field K of odd characteristic. For $\lambda^2 = -1$, consider the subgroup A of $PSL(2,K)$ generated by the elements $x = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} Z$ and $y = \begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix} Z$. The subgroup $A = \langle x, y \rangle$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. $\overline{G}$ has Lie rank $l = 1$. The order of A is 4 and $l + 1 = 2$. Therefore A is not d-abelian in $PSL_2(K)$ as A (the Sylow-2-subgroup of A) is not cyclic. In particular, A is not contained in a maximal torus of $PSL_2(K)$.

Here, one can easily observe that $C_G(A) = A$. In this case $|C_G(A)| = 4$, hence $C_G(A)$ does not contain infinitely many elements of distinct prime orders.

Remark 17. Šunkov and Kegel-Wehrfritz proved independently in [12] and [9, 10] respectively that a locally finite group satisfying minimal condition on subgroups is necessarily a Černikov group. By its definition, a Černikov group contains only finitely many elements of distinct prime orders. Hence, in Theorem 15 we proved that in a locally finite, simple group of Lie type, centralizer of a d-abelian subgroup can not be Černikov, that is, it can not satisfy minimal condition.

Centralizers in Non-Linear Simple Locally Finite Groups

For non-linear simple locally finite groups, we try to answer a different version of Hartley's question:

Question 18. *Determine all non-linear simple locally finite groups G and finite subgroups $F \leq G$ such that the centralizer of F in G contains infinitely many elements of distinct prime orders?*

Recall that the non-linear simple locally finite groups are studied by using Kegel covers, whose factors are finite simple groups:

Here, since G_i/N_i's are finite simple groups, by the Classification of Finite Simple Groups, we know that each factor is either an alternating group, or a simple group of Lie type, or a sporadic group. Since there are only finitely many sporadic groups, for any locally finite group G there exist a Kegel cover whose factors are either alternating groups or simple groups of Lie type. Indeed, for a simple locally finite group G, there are only 4 possible cases:

1. G has a Kegel cover with all G_i/N_i's are alternating groups, or,
2. G has a Kegel cover with all G_i/N_i's are a fixed type classical group with unbounded rank parameters, or,
3. G has a Kegel cover with all G_i/N_i's are a fixed type classical group with bounded rank parameters, or,
4. G has a Kegel cover with all G_i/N_i's are a fixed type exceptional groups.

In cases (3) and (4), the group is linear. So, if we have a non-linear simple locally finite group, then the Kegel cover is either alternating type or a fixed classical type with unbounded rank parameters.

Definition 19. *Let G be a countably infinite simple locally finite group and F be a finite subgroup of G. The group F is called a $\mathcal{K}$-semisimple subgroup of G, if G has a Kegel sequence $\mathcal{K} = \{(G_i, N_i) : i \in \mathbb{N}\}$ such that $(|N_i|, |F|) = 1$, N_i are soluble for all i and if G_i/N_i is a linear group over a field of characteristic p_i, then $(p_i, |F|) = 1$.*

First we considered non-linear simple locally finite groups with a Kegel sequence $\mathcal{K} = \{(G_i, N_i) : i \in \mathbb{N}\}$ where $N_i = 1$ for every i, and obtained the following result:

Theorem 20. *[3] Let G be a non-linear simple locally finite group which has a Kegel sequence $\mathcal{K} = \{(G_i, 1) : i \in \mathbb{N}\}$ consisting of finite simple subgroups. Then for any finite $\mathcal{K}$-semisimple subgroup F, the centralizer $C_G(F)$ has an infinite abelian subgroup A isomorphic to the restricted direct product of $\mathbb{Z}_{p_i}$ for infinitely many distinct primes p_i.*

We proved the following result on centralizers of $\mathcal{K}$-semisimple subgroups in non-linear simple locally finite groups with N_i are not necessarily trivial.

Theorem 21. *[3] Let G be a non-linear simple locally finite group and $\mathcal{K} = \{(G_i, N_i) \mid i \in \mathbb{N}\}$ be a Kegel sequence of G. Then for any finite $\mathcal{K}$-semisimple subgroup F, the centralizer $C_G(F)$ has a subgroup A containing elements of order p_i for infinitely many prime p_i. In particular $C_G(F)$ is infinite.*

Remark 22. Observe that without assuming "F is a $\mathcal{K}$-semisimple subgroup", the same result need not be true, as the groups constructed by Meierfrankenfeld in [11] are non-linear simple locally finite groups which contains elements whose centralizer is a p-group for some fixed prime p.

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