GROUPS WHOSE VANISHING CLASS SIZES ARE NOT DIVISIBLE BY A GIVEN PRIME

by

S. Dolfi, E. Pacifici and L. Sanus

1.- Introduction.

A well-established research area in finite group theory consists in exploring the relationship between the structure of a group G and certain sets of positive integers, which are naturally associated to G. One of those sets, denoted by $\mathrm{cs}(G)$, is the set of conjugacy class sizes of the elements of G.
A classical remark concerning the influence of $\mathrm{cs}(G)$ on the group structure of G is the following:

Theorem ([4], Theorem 33.4). *If p is a prime number which does not divide any element of $\mathrm{cs}(G)$, then G has a central Sylow p-subgroup.*

In view of that, one can ask whether particular subsets of $\mathrm{cs}(G)$ still encode nontrivial information on the structure of G. For instance, recall that an element g of G is said to be a *real element* of G if every irreducible complex character of G takes a real value on g. In [1] the following result is proved:

Theorem. *If the sizes of the conjugacy classes of real elements of G are all odd numbers, then G has a normal Sylow 2-subgroup.*

2.- Vanishing conjugacy classes.

We focus on another subset of $\mathrm{cs}(G)$, also "filtered" by the set $\mathrm{Irr}(G)$ of irreducible characters of G.

Definition. An element $g \in G$ is called a *vanishing element* of G if there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi(g) = 0$. We say that the conjugacy class of such an element is a *vanishing conjugacy class* of G.

Given a prime number p, we consider the situation in which no vanishing conjugacy class of G has size divisible by p. Since the symmetric group $\mathrm{Sym}(3)$ has only vanishing conjugacy classes of size 3, we can not expect to obtain that G has a normal Sylow p-subgroup. Nevertheless, we can prove the following.

Theorem A ([2]). Let G be a finite group, and p a prime number. If the size of every vanishing conjugacy class of G is not divisible by p, then G has a normal p-complement and abelian Sylow p-subgroups.

We would like to mention that the above result should also be compared with the following:

Theorem ([3], Corollary A). *If p is a prime number and the order of every vanishing element of G is not divisible by p, then G has a normal Sylow p-subgroup.*

3.- Examples.

Let p be a prime number and P an abelian p-group. For every choice of a p'-group K, the group $G = P \times K$ satisfies the assumptions of Theorem A. A group is of this type if and only if it has a central Sylow p-subgroup, and in this case *every* conjugacy class has size coprime to p.

It is tempting to conjecture that some further structural information can be derived for groups as in Theorem A which are not of this type. The following example shows that, in any case, we can not expect solvability.

Example. Let p and q be prime numbers such that $p \geq 7$ and $q \equiv 1 \mod 5p$ (such a q certainly exists for every choice of p, by Dirichlet's theorem on primes in an arithmetic progression). Let V be the additive group of the field $\mathrm{GF}(q^2)$. If λ is an element of order p in the multiplicative group of $\mathrm{GF}(q)$, then the scalar matrix Λ with eigenvalue λ yields a fixed-point-free automorphism of order p of V. Furthermore, since $q^2 \equiv 1 \mod 5$, there exists a subgroup K of $\mathrm{Aut}(V) \simeq \mathrm{GL}(2, \mathrm{q})$ acting fixed-point freely on V and such that $K \simeq \mathrm{SL}(2,5)$ ([5, V.8.8 b)]). Now, set $H = \langle\Lambda\rangle K$, and let G be the semidirect product $V \rtimes H$ formed according to the natural action. As $[\langle\Lambda\rangle, K] = 1$ and $(|\langle\Lambda\rangle|, |K|) = 1$, the group G is a Frobenius group with kernel V. It is not difficult to check that p does not divide the size of any vanishing conjugacy class of G.

Another natural question is whether, for a group G as in Theorem A, $G/\mathbf{Z}(G)$ is either a p'-group or a Frobenius group. Also in this case, the answer is negative.

Example. Denote by C_n a cyclic group of order n and set $H := C_5 \times \mathrm{Sym}(3)$. Consider an action of H on C_{11} whose kernel K is the Sylow 3-subgroup of H, and let $G = C_{11} \rtimes H$ be the corresponding semidirect product. It can be checked that 5 does not divide the size of any vanishing conjugacy class of G. (In fact, the sizes of the vanishing conjugacy classes of G are 11, 22 and 33.) Anyway, the centre $\mathbf{Z}(G)$ is trivial and G is not a Frobenius group.

References

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Silvio Dolfi, Università di Firenze, e-mail: dolfi@math.unifi.it
Emanuele Pacifici, Università degli Studi di Milano, e-mail: emanuele.pacifici@unimi.it
Lucia Sanus, Universitat de València, e-mail: lucia.sanus@uv.es