On finite groups whose conjugacy class sizes are in an arithmetic progression

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In [8], Huppert states that if the set of character degrees of a finite group G is $\{1, 2, 3, ...k\}$ then $k \in \{1, 2, 3, 4, 6\}$ and a description of the groups for each possible k is given; in [3] we present an analogue for conjugacy class sizes.

Our purpose here is to generalize the result for conjugacy class sizes considering the situation when the non-central conjugacy class sizes are in an arithmetic progression.

Of course one can observe that the class of finite groups G having at most two conjugacy class sizes greater than 1, studied by Ito in [5] and [6] as groups of type either $[n_1, 1]$ or $[n_2, n_1, 1]$ satisfy our assumption. In the first case it is well-known that n_1 is a $p-$power (p prime) and that $G = P \times A$ where P is a $p-$group having the same property and A is abelian, in the second case G is soluble.

Starting our discussion we will denote with $\mathcal{AP}(a, d, r)$ the class of those finite groups for which the non central conjugacy class sizes are:

$$a, a + d, a + 2d, ..., a + rd.$$

In literature one can already find papers on this subject, precisely concerning the class $\mathcal{AP}(a, 1, r)$. In [3] one proves that in any case $r \leq 1$ and one gives a classification of these groups (see Theorem 1 and 2).

Examining the list of groups of order less than 100 we observe that some groups of order 96 satysfy our assumption, in particular the non-central conjugacy class sizes are 2, 4, 6, 8. Starting from this examples we classify those finite groups whose non-central conjugacy class sizes are 2, 4, 6, 8.

Theorem 1 *Let G be a finite group with conjugacy class lengths 1, 2, 4, 6 and 8. Then $G = ((Q \rtimes C) * B) \times A$, where Q is the 8-element quaternion group, C is a cyclic group of order 3^k $(k \geq 1)$, B is a $2-$group with $|B'| = 2$, A is an abelian group, $Q \rtimes C$ is a semidirect product with C acting on Q nontrivially, and $(Q \rtimes C) * B$ is a central product with $(Q \rtimes C) \cap B = Z(Q) = B'$. Conversely, for every group G of the given structure the class lengths are exactly $1, 2, 4, 6$ and 8.*

Remark 2 *We have $|G| = 4 \cdot 3^k \cdot |B| \cdot |A|$, hence the smallest group satisfying the conditions in the Theorem has order 96 with $|C| = 3, |B| = 8, A = 1$. There are two isomorphism types of such groups of order 96, corresponding to $B \simeq Q$ and $B = D_4$, the dihedral group of order 8.*

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