Universidad del País Vasco · Euskal Herriko Unibertsitatea

Positive laws on large sets of generators and on word values

Cristina Acciarri

ISCHIA GROUP THEORY 2010

Positive words and laws on groups

- Let X be an alphabet of symbols $\{x_1, x_2, \ldots\}$; a word on X is an element of the free group on X. The word w is positive if it does not involve any inverses of the symbols x_i.
- Let T be a subset of a group G. If α, β are two different positive words on $\{x_1, x_2, \ldots, x_n\}$, T satisfies the positive law
$$\alpha(x_1, \ldots, x_n) \equiv \beta(x_1, \ldots, x_n)$$
if every substitution $x_i \mapsto t_i$ with $t_i \in T$ gives the same value for α and β as elements in G. The degree of the law is $\max\{|\alpha|, |\beta|\}$. If α and β have the same length the law is homogeneous.
- A basic example of positive laws is given by a series of homogeneous positive laws on two symbols. First, we define recursively two series of positive words on $\{x, y\}$:
$$\begin{cases} \alpha_0 = x, & \beta_0 = y, \\ \alpha_c = \alpha_{c-1}\beta_{c-1}, & \beta_c = \beta_{c-1}\alpha_{c-1}, \quad (c \geq 1). \end{cases}$$
Let $M_c(x, y)$ denote the positive law
$$\alpha_c(x, y) \equiv \beta_c(x, y).$$
It is a homogeneous positive law of degree 2^c and it is said to be a *Malcev-Thue-Morse law*. For example, $M_1(x, y)$ is $xy \equiv yx$ and $M_2(x, y)$ is $xyyx \equiv yxxy$.
- It is well known that every nilpotent group of class c satisfies $M_c(x, y)$ and, as a consequence, that every nilpotent-by-(finite exponent) group satisfies a law of the form $M_c(x^e, y^e)$.
- In 1953 A. I. Malcev conjectured that a group G satisfying a positive law should be nilpotent-by-(finite exponent) but unfortunately this conjecture is false because in 1996 Olshanskii and Storozhev constructed a counterexample.
- The good news is that the conjecture holds for many classes of groups or, more precisely, for the big class of locally graded groups (cfr. [1] and [2]).

Positive laws on "large" sets of generators

- Suppose that we do not know whether the whole group G satisfies a positive law, but only that a set of generators T of G satisfies a positive law.
- Does this imply the possibility to extend the property of satisfying a positive law to the whole group G?
- This depends on the "size" of the generating set T with respect to the whole group G.
- If the set T is sufficiently large, for example, if T is normal in G and commutator closed, i.e. closed under taking conjugates by elements of G and commutators of its elements, then G.A. Fernández-Alcober and P. Shumyatsky obtain the following positive result for a finitely generated residually-p group (cfr. [3] and [4]).

Theorem A (2007, G.A. Fernández-Alcober and P. Shumyatsky)

Let G be a d-generated residually-p group which satisfies a certain law $w \equiv 1$. Suppose that G is generated by a commutator-closed normal subset T satisfying a positive law of degree n. Then there exists a finite set of primes $P(n)$, which depends only on n, with the following property: if $p \notin P(n)$, then G is nilpotent of $\{n, p, d, w\}$-bounded class. In particular, the whole group G satisfies a positive law of $\{n, p, d, w\}$-bounded degree.

The law $M_c(x, y)$ for infinitely generated groups

- What will happen if in Theorem A we get rid of the condition of finite generation for the group G?
- We see that the result does not remain valid any more.
- By fixing the law $M_c(x, y)$ and eliminating the finitely generated condition for G, we are able to construct an example for every odd c, as it is stated in the following theorem.

Theorem B (G.A. Fernández-Alcober and C. A.)

For each $c \geq 3$, there exists an infinitely generated metabelian group G such that:
(i) *G is residually-p for all primes p.*
(ii) *G can be generated by a commutator-closed normal subset T satisfying the positive law $M_c(x, y)$.*
(iii) *G does not satisfy a positive law.*

Idea of the proof

We want to construct an infinite direct product
$$G = G_c \times G_{c+1} \times \cdots \times G_n \times \cdots$$
where every group G_n has these properties:

- G_n is a nilpotent residually-p group.
- G_n can be generated by a normal commutator-closed subset T_n which satisfies $M_c(x, y)$.
- G_n does not satisfy $M_n(x, y)$.

Note that $n \geq c$ and the "distance" between $M_c(x, y)$ and $M_n(x, y)$ increases with n.
We take $G_n = B_n \ltimes A_n$, where:

- $B_n = \langle t_1, \ldots, t_n \rangle$ is a group of matrices of size $d = d(n)$ and $t_1, \ldots, t_n$ commute.
- $A_n = \mathbb{Z} \times \overset{d(n)}{\cdots} \times \mathbb{Z}$.

We put $T_n = t_1 A_n \cup \cdots \cup t_n A_n \cup A_n$, which is commutator-closed, normal and generates all of G_n. As a consequence of a technical lemma we are able to choose the matrices $t_1, \ldots, t_n$ such that T_n satisfies the law $M_c(x, y)$ and G_n does not satisfy $M_n(x, y)$.
Note that G_n is residually-p for every prime p because it is a finitely generated torsion-free nilpotent group. Finally, by construction, we obtain that:

- G is metabelian and residually-p for every prime p.
- G can be generated by $T = \bigcup_{n \geq c} T_n$.
- T is normal commutator-closed and it satisfies $M_c(x, y)$.
- G cannot satisfy any law $M_n(x, y)$. By way of contradiction, we also prove that the group G does not satisfy a positive law.

Positive laws on normal sets: p-adic analytic case for almost all primes

- Another goal is to improve Theorem A in the particular case of a p-adic analytic group G with the weaker hypothesis that the generating set T is only normal.
- The key ingredients for this result are:
  - the analysis of the unipotent action of the normal generating set T on the abelian normal sections of G;
  - the properties of G as a p-adic analytic pro-p group.
- We obtain a positive result for almost all primes but we only need topological generation for the whole group G. We have the following theorem.

Theorem C (G.A. Fernández-Alcober and C. A.)

Let G be a p-adic analytic pro-p group of rank r which satisfies a certain law $w \equiv 1$. Suppose that G can be topologically generated by a normal subset T satisfying a positive law of degree n. If $p \notin P(n)$, then G is nilpotent of $\{p, r, n, w\}$-bounded class. In particular, the whole group G satisfies a positive law of $\{p, r, n, w\}$-bounded degree.

Positive laws on normal sets: residually-p case for almost all primes

As an application of Theorem C, by using the pro-p completion, we obtain a similar result for residually-p groups of finite upper rank.

Theorem D (G.A. Fernández-Alcober and C. A.)

Let G be a residually-p group of finite upper rank r which satisfies a certain law $w \equiv 1$. Suppose that G can be generated by a normal subset T satisfying a positive law of degree n. If $p \notin P(n)$, then G is nilpotent of $\{p, r, n, w\}$-bounded class.

Positive laws on normal sets of finite width: nilpotency for all primes

- In Theorem C the finite set $P(n)$ of "bad" primes is a real obstruction in fact it is possible to construct a powerful counterexample for every prime p.
- In order to improve the result we have to impose the extra condition of finite width on the normal generating set T. Recall that we say that T has finite width m if every element of $\langle T \rangle$ is a product of at most m elements of $T \cup T^{-1}$.
- We have the following result for a powerful p-adic analytic pro-p group.

Theorem E (G.A. Fernández-Alcober and C. A.)

Let G be a powerful pro-p group of rank r which satisfies a certain law $w \equiv 1$. Suppose that $G = \langle T \rangle$, where T is a normal generating subset of width m, which satisfies a positive law of degree n. Then G is nilpotent of $\{p, r, n, w, m\}$-bounded class. In particular, G satisfies a positive law of $\{p, r, n, w, m\}$-bounded degree.

Positive laws on word values

- The previous problem about a "large" generating set T satisfying a positive law and the nilpotency of the whole group G has an application to the case of the values of a word w.
- Let G_w be the subset of all values of w in a group G. Does a positive law on G_w imply a positive law on the corresponding verbal subgroup $w(G) = \langle G_w \rangle$?
- Note that, by definition, G_w is always a normal set but it not need to be commutator-closed.
- By applying Theorem C to the case of a verbal subgroup we obtain the following result.

Theorem F (G.A. Fernández-Alcober and C. A.)

Let G be a p-adic analytic pro-p group of rank r and let w be any word. If all w-values in G satisfy a positive law of degree n, and $p \notin P(n)$, then the verbal subgroup $w(G)$ is nilpotent of $\{p, r, n, w\}$-bounded class. In particular, $w(G)$ satisfies a positive law of $\{p, r, n, w\}$-bounded degree.

- By a result in [5] every word w has finite width in a compact p-adic analytic group, i.e the subset G_w has finite width.
- As a consequence of this fact $w(G) = \overline{w(G)}$ and so G_w generates the topological closure $\overline{w(G)}$ also in an abstract sense.
- Thus, we may apply Theorem E to a powerful verbal subgroup in a p-adic analytic pro-p group and we obtain the following result.

Theorem G (G.A. Fernández-Alcober and C. A.)

Let G be a p-adic analytic pro-p group, and let w be any word. If all w-values in G satisfy a positive law and the verbal subgroup $w(G)$ is a finitely generated powerful pro-p group, then $w(G)$ is nilpotent. In particular, $w(G)$ satisfies a positive law.

References

[1] **B. Bajorska and O. Macedońska**, On positive law problems in the class of locally graded groups, *Comm. Algebra* **32**, No. 5, (2004), 1841-1846.
[2] **R.G. Burns and Yu. Medvedev**, Groups laws implying virtual nilpotence, *J. Austral. Math. Soc.* **74** (2003), 295-312.
[3] **G.A. Fernández-Alcober and P. Shumyatsky**, Positive laws on word values in residually-p groups, preprint.
[4] **G. Fernández Alcober and P. Shumyatsky**, Positive laws on large sets of generators and on word values, *Proceedings Ischia Group Theory 2006* (2007), 125-137.
[5] **A. Jaikin-Zapirain**, On the verbal width of finitely generated pro-p groups, *Revista Matemática Iberoamericana* **168** (2008), 393-412.