Fractional Brownian motion (fBm)

$B^H = \{B^H_t, t \geq 0\}$

$H \in (0,1)$

i) $B^H_0 = 0$

ii) $E(B^H_t) = 0 \quad \forall t$

iii) $E(B^H_t B^H_s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$

$$\Rightarrow E(B^H_t)^2 = \frac{1}{2}\left(t^{2H} + t^{2H}\right) = t^{2H}$$

The process has stationary increments

$\delta > 0 \quad t \to I^H_{t,\delta} = B^H_{t+\delta} - B^H_t$ doesn't depend on t

I^H is Gaussian and centered

$$E(I^H_{t,\delta})^2 = E(B^H_{t+\delta} - B^H_t)^2 = E(B^H_{t+\delta})^2 + E(B^H_t)^2 - 2E B^H_t B^H_{t+\delta}$$
$$= (t+\delta)^{2H} + t^{2H} - \cancel{2}\cdot\frac{1}{\cancel{2}}\left(t^{2H} + (t+\delta)^{2H} - |\cancel{t}-\cancel{t}-\delta|^{2H}\right)$$
$$= \cancel{(t+\delta)^{2H}} + \cancel{t^{2H}} - \cancel{t^{2H}} - \cancel{(t+\delta)^{2H}} + |\delta|^{2H} = \delta^{2H}$$

$$E(I^H_{t,\delta} I^H_{s,\delta}) = g_\delta(|t-s|) \qquad t > s > 0, \quad t - s = \tau$$

$$2E(I^H_{t,\delta} I^H_{s,\delta}) = 2E\left(B^H_{t+\delta} - B^H_t\right)\left(B^H_{s+\delta} - B^H_s\right)$$
$$= 2E\left(B^H_{t+\delta} B^H_{s+\delta} - B^H_{t+\delta} B^H_s - B^H_t B^H_{s+\delta} + B^H_t B^H_s\right)$$
$$= 2E\left(B^H_{t+\delta} B^H_{s+\delta}\right) - 2E\left(B^H_{t+\delta} B^H_s\right) - 2E\left(B^H_t B^H_{s+\delta}\right) + 2E\left(B^H_t B^H_s\right)$$
$$= \cancel{2}\cdot\frac{1}{\cancel{2}}\left((t+\delta)^{2H} + |s+\delta|^{2H} - |t+\cancel{\delta}-s-\cancel{\delta}|^{2H}\right) - \cancel{2}\frac{1}{\cancel{2}}\left((t+\delta)^{2H} + s^{2H} - |t+\delta-s|^{2H}\right)$$
$$- \cancel{2}\cdot\frac{1}{\cancel{2}}\left(t^{2H} + (s+\delta)^{2H} - |t-s-\delta|^{2H}\right) + \cancel{2}\cdot\frac{1}{\cancel{2}}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$$
$$= \cancel{(t+\delta)^{2H}} + \cancel{(s+\delta)^{2H}} - |t-s|^{2H} - \cancel{(t+\delta)^{2H}} - \cancel{s^{2H}} + |t+\delta-s|^{2H}$$
$$- \cancel{t^{2H}} - \cancel{(s+\delta)^{2H}} + |t-s-\delta|^{2H} + \cancel{t^{2H}} + \cancel{s^{2H}} - |t-s|^{2H}$$
$$= |t+\delta-s|^{2H} - 2|t-s|^{2H} + |t-s-\delta|^{2H}$$
$$= |\tau+\delta|^{2H} - 2\tau^{2H} + |\tau-\delta|^{2H}$$

$$E(I^H_{t,\delta} I^H_{s,\delta}) = \frac{1}{2}\left(|\tau+\delta|^{2H} - 2\tau^{2H} + |\tau-\delta|^{2H}\right) = g_\delta(|t-s|)$$
$$= g_\delta(\tau)$$

Fractional Brownian motion (fBm)

$B^H = \{B^H_t, t \geq 0\}$

$H \in (0,1)$

i) $B^H_0 = 0$

ii) $E(B^H_t) = 0 \quad \forall t$

iii) $E(B^H_t B^H_s) = \frac{1}{2} \cdot \left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$

$\Rightarrow E(B^H_t)^2 = \frac{1}{2}\left(t^{2H} + t^{2H}\right) = t^{2H}$

The process has stationary increments

$\delta > 0 \quad t \to I^H_{t,\delta} = B^H_{t+\delta} - B^H_t$ doesn't depend on t

I^H is Gaussian and centered

$$E(I^H_{t,\delta})^2 = E(B^H_{t+\delta} - B^H_t)^2 = E(B^H_{t+\delta})^2 + E(B^H_t)^2 - 2E B^H_t B^H_{t+\delta}$$
$$= (t+\delta)^{2H} + t^{2H} - \cancel{2} \cdot \frac{1}{\cancel{2}}\left(t^{2H} + (t+\delta)^{2H} - |\cancel{t}-\cancel{t}-\delta|^{2H}\right)$$
$$= \cancel{(t+\delta)^{2H}} + \cancel{t^{2H}} - \cancel{t^{2H}} - \cancel{(t+\delta)^{2H}} + |\delta|^{2H} = \delta^{2H}$$

$E(I^H_{t,\delta} I^H_{s,\delta}) = g_\delta(|t-s|) \qquad t > s > 0, \quad t - s = \tau$

$$2E(I^H_{t,\delta} I^H_{s,\delta}) = 2E(B^H_{t+\delta} - B^H_t)(B^H_{s+\delta} - B^H_s)$$
$$= 2E\left(B^H_{t+\delta} B^H_{s+\delta} - B^H_{t+\delta} B^H_s - B^H_t B^H_{s+\delta} + B^H_t B^H_s\right)$$
$$= 2E(B^H_{t+\delta} B^H_{s+\delta}) - 2E(B^H_{t+\delta} B^H_s) - 2E(B^H_t B^H_{s+\delta}) + 2E(B^H_t B^H_s)$$
$$= \cancel{2} \cdot \frac{1}{\cancel{2}}\left(|t+\delta|^{2H} + |s+\delta|^{2H} - |t+\cancel{\delta}-s-\cancel{\delta}|^{2H}\right) - \cancel{2}\frac{1}{\cancel{2}}\left((t+\delta)^{2H} + |s|^{2H} - |t+\delta-s|^{2H}\right)$$
$$- \cancel{2} \cdot \frac{1}{\cancel{2}}\left(t^{2H} + (s+\delta)^{2H} - |t-s-\delta|^{2H}\right) + \cancel{2} \cdot \frac{1}{\cancel{2}}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$$
$$= \cancel{(t+\delta)^{2H}} + \cancel{(s+\delta)^{2H}} - |t-s|^{2H} - \cancel{(t+\delta)^{2H}} - \cancel{s^{2H}} + |t+\delta-s|^{2H}$$
$$- \cancel{t^{2H}} - \cancel{(s+\delta)^{2H}} + |t-s-\delta|^{2H} + \cancel{t^{2H}} + \cancel{s^{2H}} - |t-s|^{2H}$$
$$= |t+\delta-s|^{2H} - 2|t-s|^{2H} + |t-s-\delta|^{2H}$$
$$= |\tau+\delta|^{2H} - 2\tau^{2H} + |\tau-\delta|^{2H}$$

$$E(I^H_{t,\delta} I^H_{s,\delta}) = \frac{1}{2}\left(|\tau+\delta|^{2H} - 2\tau^{2H} + |\tau-\delta|^{2H}\right) = g_\delta(|t-s|)$$
$$= g_\delta(\tau)$$

$I^{H}_{t,\delta}$ is positively correlated if $H > 1/2$

$I^{H}_{t,\delta}$ is negatively correlated if $H < 1/2$.

$$E\left(I^{H}_{t,\delta}\, I^{H}_{s,\delta}\right) = \frac{1}{2}|t-s-\delta|^{2H} + \frac{1}{2}|t-s+\delta|^{2H} - |t-s|^{2H}$$

$H > 1/2$ $\quad t \geq 0 \to t^{2H}$ is convex

$$\frac{1}{2}|t-s-\delta|^{2H} + \frac{1}{2}|t-s+\delta|^{2H} \underset{\leq}{\geq} \left(\frac{|t-s-\cancel{\delta}+t-s+\cancel{\delta}|}{2}\right)^{2H} = |t-s|^{2H}$$

$$\Rightarrow \frac{1}{2}|t-s-\delta|^{2H} + \frac{1}{2}|t-s+\delta|^{2H} - |t-s|^{2H} \underset{\leq}{\geq} 0$$

$$\Rightarrow E\left(I^{H}_{t,\delta}\, I^{H}_{s,\delta}\right) \underset{\leq}{\geq} 0.$$

$H < 1/2$ $\quad t \geq 0 \to t^{2H}$ is <u>concave</u>

$$\gamma_n = E\left(B_1^H I_{n,1}^H\right), \quad n \in \mathbb{N}$$

As $n \to +\infty$ $\quad |\gamma_n| = H \cdot |2H-1|\, n^{2H-2} + O\left(n^{2H-3}\right)$

$$\sum_{n=1}^{+\infty} |\gamma_n|$$

$$I_{t,s}^H = B_{t+s}^H - B_t^H \qquad B_1^H = I_{0,1}^H = B_{0+1}^H - B_0^H = B_1^H - B_0^H = B_1^H$$

$$\gamma_n = E\left(I_{0,1}^H \, I_{n,1}\right) = \frac{1}{2}\left((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\right)$$

$$= \frac{n^{2H}}{2}\left(\left(1+\frac{1}{n}\right)^{2H} + \left(1-\frac{1}{n}\right)^{2H} - 2\right)$$

$$\boxed{(1+x)^{\alpha} = \sum_{k=1}^{+\infty} \binom{\alpha}{k} x^k \qquad \binom{\alpha}{k} := \frac{\alpha(\alpha-1)\cdot \ldots \cdot (\alpha-k+1)}{k!}}$$

$$\left(1+\frac{1}{n}\right)^{2H} = 1 + 2H \cdot \frac{1}{n} + \frac{2H(2H-1)}{2} \cdot \frac{1}{n^2} + O\left(n^{-3}\right)$$

$$\left(1-\frac{1}{n}\right)^{2H} = 1 - 2H \cdot \frac{1}{n} + \frac{2H(2H-1)}{2} \cdot \frac{1}{n^2} + O\left(n^{-3}\right)$$

$$\gamma_n = \frac{n^{2H}}{2}\left(\cancel{1} + \cancel{\frac{2H}{n}} + \frac{2H(2H-1)}{2} \cdot \frac{1}{n^2} + O(n^{-3}) + \cancel{1} - \cancel{\frac{2H}{n}} + \frac{2H(2H-1)}{2n^2} + O(n^{-3})\right) \cancel{-2}$$

$$= \frac{n^{2H}}{2}\left(\frac{2H(2H-1)}{n^2} + O\left(n^{-3}\right)\right)$$

$$= H(2H-1)\, n^{2H-2} + O\left(n^{2H-3}\right)$$

$$\sum_{n=1}^{+\infty} |\gamma_n|$$

$H \in \left(0, \frac{1}{2}\right)$ (*) $\quad H < \frac{1}{2} \Rightarrow 2H < 1 \Rightarrow 2H-2 < -1$

$H \in \left(\frac{1}{2}, 1\right)$ (*) $\quad H > \frac{1}{2} \Rightarrow 2H > 1 \Rightarrow 2H-2 > 1-2 \Rightarrow -1 < 2H-2 < 0$

(1) $\displaystyle\sum_{n=1}^{+\infty} |\gamma_n| \sim H(2H-1) \sum_{n=1}^{+\infty} n^{2H-2} < +\infty$ Short range dependence

(1) $\displaystyle\sum_{n=1}^{+\infty} |\gamma_n| \sim H(2H-1) \sum_{n=1}^{+\infty} n^{2H-2} = +\infty$ Long range dependence

Self-similarity with index H of fBm

$\cdot\ B^H_{at} \overset{d}{=} a^H B^H_t$

$0 \le t_1 < t_2 < \dots \le t_N$

$I^H_{at,as} \overset{d}{=} a^H I^H_{t,s}$

$\vec{B}^H_a := (B^H_{at_1}, \dots, B^H_{at_N})$

$\vec{\tilde{B}}^H_a := (a^H B^H_{t_1}, \dots, a^H B^H_{t_N})$

$\vec{B}^H_a \overset{d}{=} \vec{\tilde{B}}_a$

$$E \exp(i \langle \lambda, \vec{B}^H_a \rangle) = \exp\left(-\frac{1}{2} \langle \Gamma^a \lambda^T, \lambda \rangle\right)$$

$$\Gamma^a_{i,j} = E(B^H_{at_i} B^H_{at_j}), \quad i,j \le N$$

$$E \exp(i \langle \lambda, \vec{\tilde{B}}^H_a \rangle) = \exp\left(-\frac{1}{2} \langle \tilde{\Gamma}^a \lambda^T, \lambda \rangle\right)$$

$$\tilde{\Gamma}^a_{i,j} = a^{2H} E(B^H_{t_i} B^H_{t_j}), \quad i,j \le N$$

$$\Gamma^a_{i,j} = \tilde{\Gamma}^a_{i,j}, \quad i,j \le N$$

$$\Gamma^a_{i,j} = E(B^H_{at_i} B^H_{at_j}) = \frac{1}{2}\left((at_i)^{2H} + (at_j)^{2H} - |at_i - at_j|^{2H}\right)$$

$$= \frac{1}{2} a^{2H}\left((t_i)^{2H} + (t_j)^{2H} - |t_i - t_j|^{2H}\right)$$

$$= a^{2H} E(B^H_{t_i} B^H_{t_j}) = \tilde{\Gamma}^a_{i,j}$$

$$I^H_{at,as} = B^H_{at+as} - B^H_{at} \overset{d}{=} a^H (B^H_{t+s} - B^H_t) = a^H I^H_{t,s}.$$