A gentle introduction to combinatorial stochastic processes III

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Outline

Predictive probabilities

A finite Markov chain is a stochastic process in discrete time with values in a finite set $\mathcal{S}$ with the following predictive probabilities (just to fix ideas, we consider individual descriptions so that $\mathcal{S} = \{1, \ldots, g\}$)

$$\mathbb{P}(X_{m+1} = x_{m+1} | X_1 = x_1, \ldots, X_m = x_m) = \mathbb{P}(X_{m+1} = x_{m+1} | X_m = x_m). \quad (1)$$

In other words, the predictive probability only depends on the current state (*Markov property*).

Finite dimensional distributions

As a consequence of the Markov property, the finite dimensional distributions of a finite Markov chain are given by

$$p(x_1, \ldots, x_n) = \mathbb{P}(X_n = x_n | X_{n-1} = x_{n-1}) \cdots \mathbb{P}(X_2 = x_2 | X_1 = x_1) \mathbb{P}(X_1 = x_1). \quad (2)$$

If predictive probabilities do not depend on time, the Markov chain is (time) *homogeneous* and the finite dimensional distributions can be written in terms of two quantities only.

Homogeneous Markov chains I

The first quantity is the *initial probability*

$$p(x) = \mathbb{P}(X_1 = x). \tag{3}$$

The second quantity is the *transition probability*

$$P(x, x') = \mathbb{P}(X' = x' | X = x), \tag{4}$$

where X and X' represent the current value of the chain and the value of the chain at the next step, respectively. By construction, one has

$$\sum_{x' \in \{1, \ldots, g\}} P(x, x') = 1. \tag{5}$$

The convention is to represent (3) with a row vector $\mathbf{p}$ and (4) with a matrix P whose rows have non-negative entries summing up to 1 and called *stochastic matrix* (not to be confused with a random matrix).

Homogeneous Markov chains II

For any m, the predictive probabilities are given by

$$\mathbb{P}(X_{m+1} = x' | X_m = x) = P(x, x'), \tag{6}$$

and the n-point finite dimensional distributions are

$$p(x_1, \ldots, x_n) = p(x_1)P(x_1, x_2) \ldots P(x_{n-1}, x_n). \tag{7}$$

The one-point marginal distribution at step n is

$$\mathbb{P}(X_n = x_n) = \sum_{x_1, \ldots, x_{n-1}} p(x_1, \ldots, x_n) =$$

$$\sum_{x_1, \ldots, x_{n-1}} p(x_1)P(x_1, x_2) \cdots P(x_{n-1}, x_n) = \sum_{x_1} p(x_1)P^{n-1}(x_1, x_n). \tag{8}$$

The equality between the $n-1$-step transition probability and the $n-1$-th power of the one-step transition probability

$$P^{(n-1)}(x_1, x_n) = \sum_{x_2, \ldots, x_{n-1}} P(x_1, x_2) \cdots P(x_{n-1}, x_n) = P^{n-1}(x_1, x_n) \tag{9}$$

is also known as *Chapman-Kolmogorov* equation.

Example

Consider a three-state Markov chain with states labeled by $1, 2, 3$ and with initial probability

$$\mathbf{p} = (1/3, 2/3, 0)$$

and with transition probability matrix given by

$$P = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 0 & 2/3 & 1/3 \\ 1 & 0 & 0 \end{pmatrix}.$$

What is $\mathbb{P}(X_3 = 1)$, namely the probability that the matrix is in state 1 for $n = 3$? And what about $\mathbb{P}(X_{101} = 1)$?

Irreducibility

Definition (Irreducibility)

A Markov chain with finite state space is irreducible if every state can be reached from every other state.

The Markov chain of the previous example is irreducible. Here is an example of reducible Markov chain with three states: **p** whatever (it must a be a probability distribution on the initial states) and

$$Q_1 = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 2/3 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Can you think of other examples?

Invariant distribution

Definition (Invariant distribution)

A distribution π on the state space of a Markov chain with transition probability P that solves the following eigenvalue problem

$$\pi = \pi P \tag{10}$$

is called invariant distribution.

Theorem (Existence and uniqueness of the invariant distribution)

For an irreducible Markov chain on a finite state space, the solution to problem (10) *exists and is unique.*

Aperiodic Markov chains

Definition (Aperiodic state)

A state x of a Markov chain is aperiodic if $\gcd(k : \mathbb{P}(X_{k+1} = x | X_1 = x) > 0) = 1$.

Proposition

If a finite irreducible chain has an aperiodic state then all the states are aperiodic and the chain can be called aperiodic. In particular if at least one diagonal term of the transition probability matrix is non-zero, the chain is aperiodic.

The three-state chain of the example slide is aperiodic. Here is a simple example of periodic chain (again you can start with any initial probability):

$$Q_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Limit theorem

Theorem

Consider an irreducible and aperiodic Markov chain with finite state space with invariant distribution π, then for every couple of states x and x', one has

$$\lim_{n \to \infty} P^n(x, x') = \pi(x'). \tag{11}$$

In other words, for irreducible and aperiodic Markov chains with finite state space, the invariant distribution is also the *equilibrium* distribution. An immediate corollary of this result is that every irreducible and aperiodic Markov chain with finite state space converges to its stationary version.

Strong law of large numbers

Theorem

Consider an irreducible and aperiodic Markov chain with finite state space with invariant distribution π, denote by $n(x)$ the number of times state x appears in a single realization of the chain up to time n. Then for every state x, one almost surely has

$$\lim_{n\to\infty} \frac{n(x)}{n} = \pi(x). \tag{12}$$

In other words, for irreducible and aperiodic Markov chains with finite state space, the empirical frequency with which states appear in a single realization approximates the equilibrium distribution.

Meaning and applications of the two theorems

Remark (Statistical equilibrium)

Together, the limit theorem and the strong law of large number for finite irreducible and aperiodic Markov chains vindicate Boltzmann's ideas on statistical equilibrium. Indeed Paul and Tatiana Ehrenfest chose an irreducible Markov chain, the so-called Ehrenfest urn model, to illustrate Boltzmann's ideas in 1907 and later in 1911. If you are interested you can read P. Ehrenfest and T. Ehrenfest, Begriffliche Grundlagen der statistischen Auffassung in der Mechanik, in F. Klein and C. Müller (eds), Encyclopädie der Mathematischen Wissenschaften mit Einschluß ihrer Anwendungen 4, 3–90, Teubner, Leipzig (1911). English translation in P. Ehrenfest and T. Ehrenfest, The Conceptual Foundations of the Statistical Approach in Mechanics, Dover, New York (1990).

Remark (Markov chain Monte Carlo)

The limit theorem and the strong law of large numbers also lie at the foundation of Markov chain Monte Carlo (MCMC) methods.

Remark (Speed of convergence to equilibrium)

The theory of mixing times *answers questions on the speed of convergence to equilibrium (how close is $\mathbb{P}(X_n = x)$ to $\pi(x)$ as n grows), leading to several important results on* metastability. *The theory has great practical applications for MCMC algorithms and is relevant for applied sciences.*

The master equation

Proposition (Master equation)

Consider an irreducible and aperiodic Markov chain with finite state space with invariant distribution π. One has

$$\mathbb{P}(X_{n+1} = x) - \mathbb{P}(X_n = x) = \sum_{x'} [\mathbb{P}(X_n = x')P(x', x) - \mathbb{P}(X_n = x)P(x, x')]. \quad (13)$$

Detailed balance

For the stationary chain ($\mathbb{P}(X_{n+1} = x) = \mathbb{P}(X_n = x) = \pi(x)$), one has the *balance* equation

$$0 = \sum_{x'} [\pi(x')P(x', x) - \pi(x)P(x, x')]. \quad (14)$$

If *detailed balance* holds

$$\pi(x')P(x', x) = \pi(x)P(x, x'), \quad (15)$$

the chain is called *reversible*. An immediate consequence of reversibility is that a reversible irreducible Markov chain on a finite state space with symmetric transition probability (for every couple of states x and x', $P(x, x') = P(x', x)$) has the uniform distribution over the state space as invariant and equilibrium distribution.

Computing the invariant distribution

For an irreducible and aperiodic Markov chain on a finite state space one can use the following methods

1. Exact calculation from equation (10).
2. Numerical calculation using the limit theorem (11).
3. Numerical calculation using a Monte Carlo simulation of the chain and the strong law of large numbers (12).
4. If the chain is reversible, exact calculation from detailed balance (15).

The Ehrenfest urn model

Consider a system of n objects that can belong to $g = 2$ categories. In the Ehrenfest urn model, the objects are balls and they can be either in a left urn or in a right urn. The balls are numbered from 1 to n and at each step, a ball is randomly selected (e.g. by using a deck of cards numbered from 1 to n and appropriately reshuffled before every step). If the ball is on the left urn, it is moved to the right urn and vice versa. This is a periodic Markov chain of period 2. Let us first focus on individual descriptions with the left urn denoted by the label 1 and the right urn denoted by the label 0.

The Ehrenfest urn: Transition probability

Let $\mathbf{X} = \mathbf{x}$ denote the current state of the Ehrenfest urn. What is the transition probability to a state $\mathbf{X}' = \mathbf{x}'$? There are n balls and the probability of selecting one ball is n. By definition, the state $\mathbf{x}$ cannot be reached, so the diagonal terms of the $2^n \times 2^n$ transition probability matrix are zero. If $\mathbf{x}'$ is a state communicating with $\mathbf{x}$, we have

$$\mathbb{P}(\mathbf{X}' = \mathbf{x}' | \mathbf{X} = \mathbf{x}) = 1/n, \tag{16}$$

but we also have

$$\mathbb{P}(\mathbf{X} = \mathbf{x} | \mathbf{X}' = \mathbf{x}') = 1/n. \tag{17}$$

The chain is irreducible and, therefore, the invariant distribution is uniform

$$\pi(\mathbf{x}) = \frac{1}{2^n}, \ \ \forall \mathbf{x}. \tag{18}$$

The Ehrenfest urn: Example with three balls

With $n = 3$, there are $2^3 = 8$ states. Let us order the states as follows $(1,1,1)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$, $(0,0,1)$, $(0,1,0)$, $(1,0,0)$, $(0,0,0)$. The transition probability matrix is

$$Q_3 = \begin{pmatrix} 0 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 0 & 0 & 1/3 & 1/3 & 0 \\ 1/3 & 0 & 0 & 0 & 1/3 & 0 & 1/3 & 0 \\ 1/3 & 0 & 0 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 & 0 & 0 & 0 & 1/3 \\ 0 & 1/3 & 0 & 1/3 & 0 & 0 & 0 & 1/3 \\ 0 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 0 \end{pmatrix}.$$

The Ehrenfest urn: γ-delayed version

Let $\gamma \in (0,1)$ and set the diagonal term to $\mathbb{P}(\mathbf{x}|\mathbf{x}) = \gamma$ and set the non-zero off-diagonal terms to $\mathbb{P}(\mathbf{x}'|\mathbf{x}) = (1-\gamma)(1/n)$. The symmetry of the transition probability matrix is preserved, but now the chain is aperiodic. The γ-delayed version of the above example has the following transition probability matrix:

$$Q_4 = \begin{pmatrix} \gamma & \frac{1}{3}(1-\gamma) & \frac{1}{3}(1-\gamma) & \frac{1}{3}(1-\gamma) & 0 & 0 & 0 & 0 \\ \frac{1}{3}(1-\gamma) & \gamma & 0 & 0 & 0 & \frac{1}{3}(1-\gamma) & \frac{1}{3}(1-\gamma) & 0 \\ \frac{1}{3}(1-\gamma) & 0 & \gamma & 0 & \frac{1}{3}(1-\gamma) & 0 & \frac{1}{3}(1-\gamma) & 0 \\ \frac{1}{3}(1-\gamma) & 0 & 0 & \gamma & \frac{1}{3}(1-\gamma) & \frac{1}{3}(1-\gamma) & 0 & 0 \\ 0 & 0 & \frac{1}{3}(1-\gamma) & \frac{1}{3}(1-\gamma) & \gamma & 0 & 0 & \frac{1}{3}(1-\gamma) \\ 0 & \frac{1}{3}(1-\gamma) & 0 & \frac{1}{3}(1-\gamma) & 0 & \gamma & 0 & \frac{1}{3}(1-\gamma) \\ 0 & \frac{1}{3}(1-\gamma) & \frac{1}{3}(1-\gamma) & 0 & 0 & 0 & \gamma & \frac{1}{3}(1-\gamma) \\ 0 & 0 & 0 & 0 & \frac{1}{3}(1-\gamma) & \frac{1}{3}(1-\gamma) & \frac{1}{3}(1-\gamma) & \gamma \end{pmatrix}.$$

The Ehrenfest urn: Statistical description I

There are $n+1$ possible occupation states for the Ehrenfest urn. As the number of balls is fixed to n, the occupation states can be characterized by $k = 0, 1, \ldots, n$, the number of balls on the left urn. Let us consider the γ-delayed version. The occupation k can decrease by 1, increase by 1 or stay constant, if $k \neq 0$ and $k \neq n$ and we have

$$P(k, k+1) = (1-\gamma)\frac{n-k}{n},$$

$$P(k, k) = \gamma,$$

$$P(k, k-1) = (1-\gamma)\frac{k}{n}.$$

The boundary relations $P(0,0) = \gamma$, $P(0,1) = 1-\gamma$ and $P(n,n) = \gamma$, $P(n, n-1) = 1-\gamma$ complete the transition probability matrix. This is a prototypical birth-death Markov chain.

The Ehrenfest urn: Statistical description II

From (18), one immediately gets that the invariant distribution of the statistical description for the Ehrenfest urn is the binomial distribution with parameter $p = 1/2$.

$$\pi(k) = \binom{n}{k} \frac{1}{2^n}. \tag{19}$$

Brillouin's paper of 1927 I

As already discussed, the hypergeometric process corresponds to sampling without replacement from an urn whose initial composition is given by the vector **n**. A possible accommodation model amounts to choosing free positions at the instant of accommodation with uniform probability, where $\mathbf{n} = (n_1, \dots, n_g)$ is the distribution of the objects into all the categories. Brillouin gave a physical interpretation to such an accommodation model while trying to derive quantum statistics. He used a geometrical analogy and introduced the "volume" of a cell and set this volume equal to the "positive volume" of fermions, particles subject to Pauli's exclusion principle. In such an extreme case, where the volume of a cell and the volume of a particle coincide, no more than one particle can enter a cell. In Brillouin's framework a category (energy level) is made up of several cells. If the i-th energy level contains n_i cells and m_i particles, there is still room for no more than $n_i - m_i$ particles. All sequences with occupation vector different from **n** are impossible, whereas all the possible individual sequences are equiprobable. The predictive probability (conditioned on **n**) in this case is:

$$\mathbb{P}_{\mathbf{n}}(X_{m+1} = i|\mathbf{X}^{(m)}) = p_{\mathbf{n}}(i, m_i, m) = \frac{n_i - m_i}{n - m}. \tag{20}$$

Brillouin's paper of 1927 II

The urn model for the Pólya process is sampling with replacement and unitary prize from an urn whose initial composition is $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_g)$. In this case, the predictive probability is

$$\mathbb{P}_{\boldsymbol{\alpha}}(X_{m+1} = i | \mathbf{X}^{(m)}) = p_{\boldsymbol{\alpha}}(i, m_i, m) = \frac{\alpha_i + m_i}{\alpha + m} \tag{21}$$

The corresponding accommodation model is choosing each category i with probability proportional with $\alpha_i + m_i$, being α_i the initial i-th weight, and m_i the number of objects already accommodated in that category. Based on this property, Brillouin created the geometrical interpretation of "negative volume" for bosons and as many particles as available can accommodate in a cell.

The paper is: L. Brillouin, Comparaison des différentes statistiques appliquées aux problèmes de quanta, Annales de Physique, Paris, VII: 315–331 (1927).

Definition I: Generalities

Consider a population made up of n entities (agents, particles, etc.) and g categories (cells, strategies, and so on). The state of the system is described by the occupation number vector

$$\mathbf{n} = (n_1, \ldots, n_i, \ldots, n_g), \tag{22}$$

with $n_i \geq 0$, and $\sum_{i=1}^{g} n_i = n$. Therefore, the state space is the set of g-ples summing up to n; this is denoted by

$$S_g^n = \{\mathbf{n},\ n_i \geq 0,\ \sum n_i = n\}. \tag{23}$$

The dynamical discrete-time evolution is given by the sequence of random variables

$$\mathbf{Y}_0 = \mathbf{n}\,(0)\,, \mathbf{Y}_1 = \mathbf{n}\,(1)\,, \ldots, \mathbf{Y}_t = \mathbf{n}\,(t)\,, \ldots, \tag{24}$$

where $\mathbf{n}\,(t)$ belongs to S_g^n. It is assumed that equation (24) decribes the realization of a homogeneous Markov chain, whose one-step transition probability is a matrix whose entries are given by

$$P\left(\mathbf{n}, \mathbf{n}'\right) = \mathbb{P}(\mathbf{Y}_{t+1} = \mathbf{n}' | \mathbf{Y}_t = \mathbf{n}), \tag{25}$$

with $\mathbf{n}', \mathbf{n} \in S_g^n$, not depending on time explicitly. How many are the entries of this stochastic matrix?

Definition II: Unary moves I

The state of the system changes from $\mathbf{Y}_t = \mathbf{n}$ to $\mathbf{Y}_{t+1} = \mathbf{n}_i^k$, where $\mathbf{n} = (n_1, \ldots, n_i, \ldots, n_k, \ldots, n_g)$ denotes the initial state and $\mathbf{n}_i^k = (n_1, \ldots, n_i - 1, \ldots, n_k + 1, \ldots, n_g)$ the final state. This transition can be split into two distinct parts. The first one (the so-called "Ehrenfest's term") is the destruction of an entity belonging to the i-th category, with probability

$$\mathbb{P}(\mathbf{n}_i|\mathbf{n}) = \frac{n_i}{n} \tag{26}$$

The second part (the so-called "Brillouin's term") is the creation of a particle in the k-th category given the vector $\mathbf{n}_i = (n_1, n_i - 1, \ldots, n_k, \ldots, n_g)$, whose probability is:

$$\mathbb{P}(\mathbf{n}_i^k|\mathbf{n}_i) = \frac{\alpha_k + n_k - \delta_{k,i}}{\alpha + n - 1} \tag{27}$$

with $\alpha = \sum_{i=1}^{g} \alpha_i$, and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_g)$ is a vector of parameters; $\delta_{k,i}$ is the usual Kronecker's delta equal to 1 for $k = i$ and zero otherwise.

Definition III: Unary moves II

The meaning of α_i is related to the probability of accommodation on the category i if it is empty. Two case are interesting: all $\alpha_i > 0$, and all α_i negative integers. In this second case, the size of the population is limited by the inequality $n \leq |\alpha|$, and categories with occupation numbers $n_i \geq |\alpha_i|$ cannot accommodate further elements.
The resulting transition probability is:

$$\mathbb{P}(\mathbf{n}_i^k|\mathbf{n}) = \mathbb{P}(\mathbf{n}_i|\mathbf{n})\mathbb{P}(\mathbf{n}_i^k|\mathbf{n}_i) = \frac{n_i}{n}\frac{\alpha_k + n_k - \delta_{k,i}}{\alpha + n - 1}. \tag{28}$$

When all the α_i's are positive, if one starts from a given occupation vector $\mathbf{n}$, by repeated applications of (28), each possible vector of S_g^n can be reached with positive probability. Remember that the cardinality of the state space is

$$\#S_g^n = \binom{n+g-1}{n}. \tag{29}$$

Invariant distribution and statistical equilibrium

The Ehrenfest-Brillouin chain is irreducible and aperiodic. We can try and derive the invariant (and equlibrium) distribution using detailed balance. In fact, here, nothing is “irreversible” in the transition mechanism. Then, the search for the invariant distribution can be done by means of detailed balance conditions. One finds that

$$\frac{\pi(\mathbf{n}_i^k)}{\pi(\mathbf{n})} = \frac{n_i}{n_k + 1} \frac{\alpha_k + n_k}{\alpha_j + n_j - 1}, \tag{30}$$

a relationship which is satisfied by the generalized g-dimensional Pólya distribution

$$\pi(\mathbf{n}) = \mathrm{Polya}(\mathbf{n}; \boldsymbol{\alpha}) = \frac{n!}{\alpha^{[n]}} \prod_{i=1}^{g} \frac{\alpha_i^{[n_i]}}{n_i!} \tag{31}$$

with $\sum_{i=1}^{g} n_i = n$ and $\sum_{i=1}^{g} \alpha_i = \alpha$. If one has that $\{\alpha_i > 0\}_{i=1}^{g}$, then creations are positively correlated with occupation numbers $\mathbf{n}$.

Ehrenfest Brillouin model and quantum statistics: BE

A simple sub-case of equation (31) is the so-called *Bose-Einstein distribution*, which is the symmetric g-dimensional Pólya distribution with $\alpha_i = 1$, and $\alpha = g$. In this case $\alpha_i^{[n_i]}$ becomes $n_i!$, and

$$\frac{n!}{\alpha^{[n]}} = \frac{n!}{g^{[n]}}, \tag{32}$$

so that one eventually finds

$$\pi_{\mathrm{BE}}(\mathbf{n}) = \mathrm{Polya}(\mathbf{n}; \mathbf{1}) = \binom{n+g-1}{n}^{-1}, \tag{33}$$

where $\boldsymbol{\alpha} = \mathbf{1} = (1, \ldots, 1)$ is a vector containing g values equal to 1. In other words, if $\boldsymbol{\alpha} = \mathbf{1}$, all the occupation vectors have the same equilibrium probability, that is π_{BE} does not depend on $\mathbf{n}$.

Ehrenfest Brillouin model and quantum statistics: FD

When $\{\alpha_i < 0\}$ and α_i are negative integers, creations are negatively correlated with the occupation numbers $\mathbf{n}$. The equilibrium probability (31) becomes the g-dimensional hypergeometric distribution, that can be re-written as

$$\pi(\mathbf{n}) = \mathrm{HD}\left(\mathbf{n} | |\alpha|, |\boldsymbol{\alpha}|\right) = \frac{\prod_{i=1}^{g} \binom{|\alpha_i|}{n_i}}{\binom{|\alpha|}{n}}, \qquad (34)$$

with $\sum_{i=1}^{g} n_i = n$, and $n_i \leq |\alpha_i|$. A remarkable example of equation (34) is the so-called *Fermi-Dirac distribution*, which is obtained for $\alpha_i = -1$, and $\alpha = -g$. Given that $n_i \in \{0, 1\}$, the numerator is 1, and one finds that

$$\pi_{\mathrm{FD}}(\mathbf{n}) = \mathrm{Polya}(\mathbf{n}; -\mathbf{1}) = \binom{g}{n}^{-1}. \qquad (35)$$

which is uniform on all occupation vectors with $n_i \in \{0, 1\}$, as also π_{FD} does not depend on $\mathbf{n}$. Note that both π_{BE} and π_{FD} are the only uniform distributions on the occupation vectors in the two possible domains of $\boldsymbol{\alpha}$. They are at the basis of the statistical description of quantum particles.

Ehrenfest Brillouin model and quantum statistics: MB

The correlation is large and positive for small $\alpha > 0$, it is large and negative for small (integer) $\alpha < 0$. The smaller possible value of $\alpha < 0$ is $\alpha = -g$, where no more than one unit can occupy each category (this is known in Physics as *Pauli's principle*). Correlations vanish for $|\alpha| \to \infty$ from both sides, where the creation probability becomes $p_k = \alpha_k/\alpha$ independent of $\mathbf{n}$. In this limit, equation (31) becomes

$$\pi(\mathbf{n}) = \mathrm{Polya}(\mathbf{n}; \pm\infty) = \mathrm{MD}(\mathbf{n}|n, \mathbf{p}) = \frac{n!}{\prod_{i=1}^{g} n_i!} \prod_{i=1}^{g} p_i^{n_i} \tag{36}$$

which is the multinomial distribution; if $p_1 = p_2 = \ldots = p_n = 1/g$, one gets the symmetric mutinomial distribution

$$\pi_{\mathrm{MB}}(\mathbf{n}) = \frac{n!}{\prod_{i=1}^{g} n_i!} g^{-n}; \tag{37}$$

in statistical Physics, this is known as the *Maxwell-Boltzmann distribution*. The case $|\alpha| \to \infty$ coincides with an independent accommodation of the moving element; it was discussed earlier. The equilibrium probability (37) is such that all the individual descriptions $\mathbf{X}$ are equiprobable, and this distribution lies at the foundation of classical Statistical Mechanics.

It is interesting to observe that all the above distributions can be discussed in a unified way within the same probabilistic scheme, without introducing notions as "(in-)distinguishable"particles or elements, which sometimes are used even in Probability (by Feller for instance) and in Economics. The different behaviour of the microscopic entities is due to the vector parameter $\boldsymbol{\alpha}$ leading to different kinds of inter-element correlations.

Exercise 1: The Ehrenfest urn

Consider the aperiodic Ehrenfest urn model, set $n = 4$ and further consider a double selection, followed by a double independent creation. A transition path is D_1, D_2, C_1, C_2 where D_1, D_2 are conditioned on the starting state $k = 0, \ldots, 4$, whereas C_1, C_2 are independent. Therefore, one has $\mathbb{P}(C_1, C_2) = 1/4$ uniformly on all the four possibilities. Show that the transition matrix with $m = 2$ (binary moves) is different from the square of the transition matrix with $m = 1$ (unary moves) given above.

Exercise 2: The Ehrenfest Brillouin model

Write a Monte Carlo program for the Ehrenfest Brillouin model with unary moves. In the dichotomous case, compare the results for $\alpha_1 = \alpha_2 = 10$ and $\alpha_1 = \alpha_2 = 1/2$.