Stochastic integration and stochastic differential equations involving fBm

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Pathwise stochastic integration in the fractional Sobolev type spaces

Let us construct pathwise integrals $\int_0^T f(t)dB_t^H$ for stochastic process f belnging a.s. to the fractional Sobolev type spaces $I_{0+}^{\alpha}(L^p)$ with some $p > 1$. This approach was developed by Zähle [Zah98a], [Zah99], [Zah01]. Consider two non-random functions f and g defined on some interval $[a, b] \subset \mathbb{R}$, suppose that the limits $f(u+) := \lim_{\delta \downarrow 0} f(u + \delta)$, $g(u-) := \lim_{\delta \downarrow 0} g(u - \delta)$, $a \leq u \leq b$, exist, and put $f_{a+}(x) := (f(x) - f(a+))\mathbf{1}_{(a,b)}(x)$, $g_{b-}(x) := (g(b-) - g(x))\mathbf{1}_{(a,b)}(x)$. Suppose that $f_{a+} \in I_{a+}^{\alpha}(L_p[a, b])$, $g_{b-} \in I_{b-}^{1-\alpha}(L_q[a, b])$ for some $p \geq 1, q \geq 1, 1/p + 1/q \leq 1$, $0 \leq \alpha \leq 1$. Then, evidently, $D_{a+}^{\alpha} f_{a+} \in L_p[a, b]$, $D_{b-}^{1-\alpha} g_{b-} \in L_q[a, b]$ and it is possible to give the following definition.

Definition 1

The generalized (fractional) Lebesgue-Stieltjes integral $\int_a^b f(x)dg(x)$ is defined as

$$\int_a^b f(x)dg(x) := \int_a^b (D_{a+}^{\alpha} f_{a+})(x)(D_{b-}^{1-\alpha} g_{b-})(x)dx + f(a+)(g(b-)-g(a+)).$$

Lemma 2

Definition 1 does not depend on the possible choice of α.

Let $\alpha p < 1$. Then $f_{a+} \in I_{a+}^{\alpha}(L_p[a,b])$ if and only if $f \in I_{a+}^{\alpha}(L_p[a,b])$ and in this case

$$\int_a^b f(x)dg(x) = \int_a^b \left((D_{a+}^{\alpha} f)(x) - \frac{1}{\Gamma(1-\alpha)} \cdot \frac{f(a+)}{(x-a)^{\alpha}} \right) (D_{b-}^{1-\alpha} g_{b-})(x)dx$$
$$+ f(a+)(g(b-) - g(a+)) = \int_a^b (D_{a+}^{\alpha} f)(x)(D_{b-}^{1-\alpha} g_{b-})(x)dx$$
$$- f(a+) I_{b-}^{1-\alpha}(D_{b-}^{1-\alpha} g)(a) + f(a+)(g(b-) - g(a+))$$
$$= \int_a^b (D_{a+}^{\alpha} f)(x)(D_{b-}^{1-\alpha} g_{b-})(x)dx. \tag{1}$$

Lemma 3

Let $g_{b-} \in I_{b-}^{1-\alpha}(L_p[a,b]) \cap C[a,b]$ for some $p > \frac{1}{1-\alpha}$, $0 < \alpha < 1$. Then for any $a < c < d < b$

$$\int_a^b (D_{a+}^{\alpha} \mathbf{1}_{[c,d)})(x)(D_{b-}^{1-\alpha} g_{b-})(x)dx = g(d) - g(c). \tag{2}$$

Corollary 4

Let the function g is Hölder continuous with some exponent $0 < \lambda \leq 1$*, then* $g_{b-} \in I_{b-}^{1-\alpha}(L_p[a,b])$ *for any* $p \geq 1$ *and* $1 - \alpha < \lambda$*. So, we can take* $p > 2/\lambda$, $\alpha = 1 - \lambda/2$ *and obtain* (2).

Corollary 5

For any step function $f_\pi(x) = \sum_{k=0}^{n-1} c_k \mathbf{1}_{[x_k, x_{k+1})}(x)$ *with* $a = x_0 < \ldots < x_n = b$ *and g satisfying the conditions of Lemma 3, we have that* $\int_a^b f(x)dg(x) = \sum_{k=0}^{n-1} c_k(g(x_{k+1}) - g(x_k))$.

Further we suppose that $g(b-) = g(b)$ and $g(a+) = g(a)$.
Denote $BV[a,b]$ the class of functions of bounded variation on $[a,b]$.

Lemma 6

Let the functions $f_{a+} \in I^{\alpha}_{a+}(L_p[a,b])$, $g_{b-} \in I^{1-\alpha}_{b-}(L_q[a,b]) \cap BV[a,b]$ *with* $p \geq 1, q \geq 1, 1/p + 1/q \leq 1$ *and*

$$\int_a^b I^{\alpha}_{a+}(|(D^{\alpha}_{a+} f)|)(x)|g|(dx) < \infty. \tag{3}$$

Then

$$\int_a^b f(x)dg(x) = \text{(L-S)} \int_a^b f(x)dg(x).$$

Now we consider the case of Hölder functions f and g. The existence of (R-S) $\int_a^b fdg$ for $f \in C^\lambda[a,b]$, $g \in C^\mu[a,b]$ with $\lambda + \mu > 1$ was established by Kondurar [Kon37]. Moreover, this integral coincides with $\int_a^b fdg$, as the next theorem states.

Let $f \in C^\lambda[a,b]$ for some $0 < \lambda \le 1$, $|f(x) - f(y)| \le c(\lambda)|x-y|^\lambda$, $x, y \in [a,b]$. Consider the following step function

$$f_\pi(x) = \sum_{k=0}^{n-1} f(x_k)\mathbf{1}_{[x_k, x_{k+1})}(x),$$

where the partition $\pi = \{a = x_0 < x_1 < \ldots < x_n = b\}$.
Evidently, $\lim_{|\pi| \to 0} \sup_\pi \|f_\pi - f\|_{L_\infty[a,b]} = 0$.

Theorem 7

1) For any $0 < \alpha < \lambda$

$$\lim_{|\pi| \to 0} \sup_\pi \|(D^\alpha_{a+} f_\pi) - (D^\alpha_{a+} f)\|_{L_1[a,b]} = 0.$$

2) Let $f \in C^\lambda([a,b])$, $g \in C^\mu[a,b]$ *with* $\lambda + \mu > 1$ *then* $(\mathrm{R\text{-}S}) \int_a^b f dg$ *exists and*

$$\int_a^b f dg = (\mathrm{R\text{-}S}) \int_a^b f dg.$$

Now we establish the properties of generalized integral $\int_s^t fdg$ as the function of upper and lower boundaries.

Lemma 8 ([Zah98a])

1) Let $a \le s < t \le b$ and the functions f and g satisfy the assumptions (i) $(f \cdot \mathbf{1}_{(s,t)}) \in I_+^{\alpha}(L_p[a,b])$, $g_{b-} \in I_-^{1-\alpha}(L_q[a,b])$ for some $0 < \alpha < 1, p \ge 1, q \ge 1, 1/p + 1/q \le 1$, (ii) $f_{s+} \in I_+^{\alpha'}(L_{p'}[s,t])$, $g_{t-} \in I_-^{1-\alpha'}(L_{q'}[s,t])$ for some $0 < \alpha' < 1, p' \ge 1, q' \ge 1, 1/p' + 1/q' \le 1$. Then

$$\int_a^b \mathbf{1}_{(s,t)} fdg = \int_s^t fdg.$$

2) The equality

$$\int_s^t fdg + \int_t^u fdg = \int_s^u fdg$$

holds for $a \le s < t < u \le b$, if all the integrals exist as generalized Lebesgue–Stieltjes integrals.

Pathwise stochastic integration in fractional Besov type spaces

Consider the approach to pathwise stochastic integration in fractional Besov type spaces, introduced by Nualart and Răşcanu [NR00] (see also the paper of Cieselski[CKR93]).

Consider the following functional spaces. Let for $0 < \alpha < 1$ $\varphi_f^\alpha(t) := |f(t)| + \int_0^t |f(t) - f(s)|(t-s)^{-\alpha-1} ds$, $W_0^\alpha = W_0^\alpha[0\,T]$ be the space of real-valued measurable functions $f : [0, T] \to \mathbb{R}$ such that

$$\|f\|_{0,\alpha} := \sup_{t\in[0,T]} \varphi_f^\alpha(t) < \infty;$$

$W_1^\alpha = W_1^\alpha[0, T]$ be the space of real-valued measurable functions $f : [0, T] \to \mathbb{R}$ such that

$$\|f\|_{1,\alpha} := \sup_{0\le s<t\le T} \left(\frac{|f(t) - f(s)|}{(t-s)^\alpha} + \int_s^t \frac{|f(u) - f(s)|}{(u-s)^{1+\alpha}} du \right) < \infty,$$

and

$W_2^\alpha = W_2^\alpha[0, T]$ be the space of real-valued measurable functions $f : [0, T] \to \mathbb{R}$ such that

$$\|f\|_{2,\alpha} := \int_0^T \frac{|f(s)|}{s^\alpha} ds + \int_0^T \int_0^s \frac{|f(s) - f(u)|}{(s-u)^{\alpha+1}} duds < \infty.$$

The spaces $W_i^\alpha, i = 0, 1, 2$ are Banach spaces with respect to corresponding norms.

Also, for any $0 < \varepsilon < \alpha \wedge (1 - \alpha)$

$$C^{\alpha+\varepsilon}[0, T] \subset W_i^\alpha[0, T] \subset C^{\alpha-\varepsilon}[0, T], \quad i = 0, 1,$$
$$C^{\alpha+\varepsilon}[0, T] \subset W_2^\alpha[0, T].$$

Therefore the trajectories of fBm B^H with Hurst index H for a.a. $\omega \in \Omega$ belong to $W_1^\alpha[0, T]$ for any $T > 0$, $0 < \alpha < H$ and $\|B^H\|_{1,\alpha} < \infty$ for any $0 < \alpha < H$.

Moreover, for $g \in W_1^\alpha[0, T]$ its restriction to $[0, t] \subset [0, T]$ belongs to $I_-^\alpha(L_\infty[0, t])$ and

$$\Lambda_\alpha(g) := \sup_{0 \le s < t \le T} |(D_{t-}^\alpha g)_{t-}(s)| \le \frac{1}{\Gamma(1-\alpha)} \|g\|_{1,\alpha} < \infty.$$

The restriction of $f \in W_2^\alpha[0, T]$ to $[0, t] \subset [0, T]$ belongs to $I_+^\alpha(L_1[0, t])$.

Now, let $f \in W_2^{\alpha}[0,T]$, $g \in W_1^{1-\alpha}[0,T]$. Then for any $0 < t \leq T$ there exists Lebesgue integral $\int_0^t (D_{0+}^{\alpha} f)(x)(D_{t-}^{1-\alpha} g_{t-})(x)dx$, so we can define $\int_0^t fdg$ according to Definition 1 and formula (2).Moreover, for any $0 < t \leq T$ $\int_0^t fdg = \int_0^T 1_{(0,t)} fdg$, and the integral $\int_0^t fdg$ admits an estimate

$$|\int_0^t fdg| \leq \int_0^t |(D_{0+}^{\alpha} f)(x)||(D_{t-}^{1-\alpha} g_{t-})(x)|dx \\ \leq \Lambda_{1-\alpha}(g)\|f\|_{2,\alpha} \leq (\Gamma(1-\alpha))^{-1}\|g\|_{1,1-\alpha}\|f\|_{2,\alpha}.$$

Stochastic differential equations driven by fractional Brownian motion with pathwise integrals

Consider the function $\sigma = \sigma(t, x) : [0, T] \times \mathbb{R} \to \mathbb{R}$ satisfying the assumptions: σ is differentiable in x, there exist $M > 0$, $0 < \beta, \kappa \leq 1$ and for any $R > 0$ there exists $M_R > 0$ such that
(i) σ is Lipschitz continuous in x:

$$|\sigma(t,x) - \sigma(t,y)| \leq M|x-y|, \quad \forall t \in [0, T], x, y \in \mathbb{R},$$

and is of sublinear growth: there exist $\gamma \in [0,1]$ and $K_0 > 0$ such that

$$|\sigma(t,x)| \leq K_0(1 + |x|^{\gamma})$$

for all t, x; (ii) x-derivative of σ is local Hölder continuous in x:

$$|\sigma_x(t,x) - \sigma_x(t,y)| \leq M_R|x-y|^{\kappa}, \quad \forall |x|, |y| \leq R, t \in [0, T];$$

(iii) σ and x-derivative of σ are Hölder continuous in time:

$$|\sigma(t,x) - \sigma(s,x)| + |\sigma_x(t,x) - \sigma_x(s,x)| \leq M|t-s|^{\beta}, \quad \forall x \in \mathbb{R}, t, s \in [0, T].$$

Let the function $b = b(t,x) : [0, T] \times \mathbb{R} \to \mathbb{R}$ satisfy the assumptions
(iv) for any $R \geq 0$ there exists $L_R > 0$ such that

$$|b(t,x) - b(t,y)| \leq L_R|x - y|, \quad \forall |x|, |y| \leq R, \forall t \in [0, T];$$

(v) there exists the function $b_0 \in L_\rho[0, T]$ and $L > 0$ such that

$$|b(t,x)| \leq L|x| + b_0(t), \quad \forall (t,x) \in [0, T] \times \mathbb{R}.$$

Now, consider SDE with fBm B_t^H, $H \in (1/2, 1)$ on a complete probability space $(\Omega, \mathcal{F}, P)$:

$$X_t = X_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s^H, \quad t \in [0, T]. \tag{4}$$

We can now state the following result.

Theorem 9

Let the coefficients b and σ satisfy (i)–(v) with $\rho = (1 - H)^{-1}$, $\beta > 1 - H, \kappa > H^{-1} - 1$ (the constants M, M_R, R, L_R and the function b_0 can depend on ω).
Then there exists the unique solution $\{X_t, t \in [0, T]\}$ of equation (4), $X \in L_0(\Omega, \mathcal{F}, P, W_0^{1-H}[0, T])$ with a.a.trajectories from $C^{H-\varepsilon}[0, T]$ for any $0 < \varepsilon < H$.

Remark 1

Theorem 9 admits evident generalization to multidimensional case. Consider the equation on $\mathbb{R}^d$

$$X_t^i = X_0^i + \int_0^t b_i(s, X_s)ds + \sum_{j=1}^m \int_0^t \sigma_{ji}(s, X_s)dB_s^{H_j}, \quad 1 \le i \le d, t \in [0, T], \tag{5}$$

where the processes B^{H_j} are fBm's with Hurst index $H_j \in (1/2, 1), 1 \le j \le m$. Denote $\sigma = (\sigma_{ji})_{i,j=1}^{d,m}$ the matrix of "diffusions" and $b = (b_i)_{i=1}^d$ the "drift" vector, $|\sigma| := (\sum_{i,j} |\sigma_{j,i}|^2)^{1/2}$, $|b| := (\sum_i (b_i)^2)^{1/2}$, and suppose that assumptions (i)–(v) hold with these notations, $H = \min_{1 \le i \le m} H_i$, $\rho = (1 - H)^{-1}$, $\beta > 1 - H$, $\kappa > H^{-1} - 1$. Then there exists the unique vector solution X_t of equation (5) on $[0, T]$ in $L_0(\Omega, \mathcal{F}, P, W_0^{1-H}[0, T])$ with a.a. trajectories from $C^{H-\varepsilon}[0, T]$ for any $0 < \varepsilon < H$.

Theorem 9 and other similar results are establish by the standard fixed point theorem. What is the specific moment here? To establish that the respective map is a contraction, it is necessary to consider the difference of generalized Lebesgue-Stieltjes integrals of two functions, and so the difference of two fractional derivatives that in turn includes the difference of the function in two points s and t, say, appears. We get the value of the form

$$\sigma(t,x) - \sigma(t,y) - \sigma(s,x) + \sigma(s,y).$$

To deal with such double-difference, there is no another methods except to increase the smoothness of the diffusion.
Consider one example where this condition evidently holds and no additional assumption is necessary.

Some properties of OU process involving fBm

Consider the process that is a unique solution of the Langevin stochastic differential equation

$$dX_t = \theta X_t dt + \sigma dB_t^H,\ t \geq 0,\ X\big|_{t=0} = X_0, \tag{6}$$

where $\theta \in \mathbb{R}$, $\sigma > 0$. This process is called the fractional Ornstein -Uhlenbeck process and for any $H \in (0,1)$ it admits an explicit representation

$$X_t = e^{\theta t}\left(X_0 + \sigma \int_0^t e^{-\theta s} dB_s^H\right).$$

If $H \in (0, 1/2)$, then integral $\int_0^t e^{-\theta s} dB_s^H$ we understand as the result of the integration by parts:

$$\int_0^t e^{-\theta s} dB_s^H = e^{-\theta t} dB_t^H + \theta \int_0^t e^{-\theta s} B_s^H ds,$$

since the exponent has a bounded variation.

It is ergodic if $\theta < 0$ and non-ergodic if $\theta > 0$. Therefore a question appears: how to test the hypothesis concerning the sign of θ?The interest to this problem is also connected with the stability properties of the solution of the equation (6), which also depend on the sign of θ.

Simple test for the testing the hypothesis $H_0\colon \theta \le 0$ against the alternative $H_1\colon \theta > 0$

Let us propose comparatively simple test for the testing the hypothesis $H_0\colon \theta \le 0$ against the alternative $H_1\colon \theta > 0$. The main advantage of our approach is that it can be used for any $H \in (0,1)$. Moreover, the test is based on the observation of the process X at one point, therefore it is applicable for both continuous and discrete cases. The distribution of the test statistics is computed explicitly, and the power of test can be found numerically for any given simple alternative. Also we consider the hypothesis testing $H_0\colon \theta \ge \theta_0$ against $H_1\colon \theta \le 0$, where $\theta_0 \in (0,1)$ is some fixed number. Unfortunately, our approach does not enable to test the hypothesis $H_0\colon \theta = 0$ against the two-sided alternative $H_1\colon \theta \ne 0$.

For the hypothesis testing of the sign of the parameter θ we construct a test based on the asymptotic behavior of the random variable

$$Z(t) = \frac{\ln^{+} \ln |X_t|}{\ln t}, t > 1. \tag{7}$$

The following result explains the main idea. It is based on the different asymptotic behavior of Ornstein–Uhlenbeck process with positive and negative drift parameter. But we need in some auxiliary results.

One-dimensional distributions of the Ornstein–Uhlenbeck process

Lemma 10

The random variable X_t has a normal distribution $\mathcal{N}\left(X_0 e^{\theta t}, v(\theta, t)\right)$, where

$$v(\theta, t) = H \int_0^t s^{2H-1} \left(e^{\theta s} + e^{\theta(2t-s)} \right) ds. \tag{8}$$

The asymptotical behavior of the function $v(\theta, t)$ as $t \to \infty$ is as follows.

Lemma 11

1. *If* $\theta > 0$, *then* $v(\theta, t) \sim \frac{H\Gamma(2H)}{\theta^{2H}} e^{2\theta t}$, $t \to \infty$.
2. *If* $\theta < 0$, *then* $v(\theta, t) \to \frac{H\Gamma(2H)}{(-\theta)^{2H}}$, $t \to \infty$.
3. $v(0, t) = t^{2H}$, $t \geq 0$.

Almost sure limits and bounds for the Ornstein–Uhlenbeck process

Lemma 12

For $\theta > 0$

$$e^{-\theta t} X_t \to \xi_\theta \quad a.\,s. \;\; as \; t \to \infty,$$

where $\xi_\theta = X_0 + \theta \int_0^\infty e^{-\theta s} B_s^H \, ds \simeq \mathcal{N}\left(X_0, \frac{H\Gamma(2H)}{\theta^{2H}}\right)$.

Lemma 13 ([11])

There exists a nonnegative random variable ζ *such that for all* $t > 0$ *the following inequalities hold true:*

$$\sup_{0 \le s \le t} \left| B_s^H \right| \le \left(1 + t^H \ln^2 t \right) \zeta, \tag{9}$$

for $\theta > 0$

$$\sup_{0 \le s \le t} |X_s| \le \left(e^{\theta t} + t^H \ln^2 t \right) \zeta, \tag{10}$$

while for $\theta \le 0$

$$\sup_{0 \le s \le t} |X_s| \le \left(1 + t^H \ln^2 t \right) \zeta. \tag{11}$$

Moreover, ζ *has the following property: there exists* $C > 0$ *such that* $E \exp\{x\zeta^2\} < \infty$ *for any* $0 < x < C$.

Lemma 14

The value of $Z(t)$ converges a. s. to 1 for $\theta > 0$, and to 0 for $\theta \leq 0$.

Proof.

For $\theta > 0$ Lemma 12 implies the convergence

$$\ln |X_t| - \theta t \to \ln |\xi_\theta| \quad \text{a. s. as } t \to \infty,$$

where ξ_θ is a Gaussian random variable, hence, $0 < |\xi_\theta| < \infty$ a. s. Therefore,

$$\frac{\ln |X_t|}{t} \to \theta \quad \text{a. s. as } t \to \infty,$$

whence the a. s. convergence $Z(t) \to 1$ follows.
For $\theta \leq 0$ it follows from (11) that

$$\frac{|X_t|}{t} \to 0 \quad \text{a. s. as } t \to \infty.$$

Then $Z(t) \to 0$ a. s. as $t \to \infty$. □

Lemma 15

For $t > 1$ *the probability* $g(\theta, x_0, t, c) = P(Z(t) < c)$ *is given by*

$$g(\theta, X_0, t, c) = \Phi\left(\frac{e^{t^c} - X_0 e^{\theta t}}{\sqrt{v(\theta, t)}}\right) + \Phi\left(\frac{e^{t^c} + X_0 e^{\theta t}}{\sqrt{v(\theta, t)}}\right) - 1, \qquad (12)$$

where $v(\theta, t)$ *is a variance of* X_t*, and* g *is a decreasing function of* $\theta \in \mathbb{R}$*.*

Lemma 16

Let $\alpha \in (0, 1)$*. Then there exists* $t_0 > 1$ *such that for all* $t > t_0$ *there exists a unique* $c_t \in (0, 1)$ *such that* $g(0, X_0, t, c_t) = 1 - \alpha$*, and* $c_t \to 0$ *as* $t \to \infty$*.*

The constant t_0 *can be chosen as the largest* $t > 1$ *that satisfies at least one of the following two equalities*

$$g(0, X_0, t, 0) = 1 - \alpha \quad \text{or} \quad g(0, X_0, t, 1) = 1 - \alpha.$$

Testing the hypothesis $H_0\colon \theta \le 0$ against $H_1\colon \theta > 0$

We consider the test with the following procedure of testing the hypothesis $H_0\colon \theta \le 0$ against the alternative $H_1\colon \theta > 0$. For a given significance level α, and for sufficiently large value of t we choose a threshold $c = c_t \in (0,1)$, see Lemma 16. Further, when $Z(t) \le c$ the hypothesis H_0 is accepted, and when $Z(t) > c$ it is rejected. Below we will propose a technically simpler version of this test, without the computation of the threshold c, see Algorithm 1.

By Lemma 15 for a threshold $c \in (0,1)$ and $t > 1$ the probability of a type I error equals

$$\sup_{\theta \le 0} P(Z(t) \ge c) = 1 - g(0, X_0, t, c).$$

Therefore, for a significance level α we determine c_t as a solution of the equation

$$g(0, X_0, t, c_t) = 1 - \alpha. \tag{13}$$

Lemma 16 shows that for any $\alpha \in (0,1)$ it is possible to choose a sufficiently large t, such that $c_t \in (0,1)$.

Since the function $g(0, X_0, t, c)$ is strictly increasing with respect to c for $t > 1$, we see that the inequality $Z(t) \le c_t$ is equivalent to the inequality $g(0, X_0, t, Z(t)) \le g(0, X_0, t, c_t) = 1 - \alpha$. Therefore, we do not need to compute the value of c_t. It is sufficient to compare $g(0, X_0, t, Z(t))$ with the level $1 - \alpha$.

Algorithm 1 The hypothesis $H_0 : \theta \le 0$ against the alternative $H_1 : \theta > 0$ can be tested as follows.

1. Find t_0 defined in Lemma 16. The algorithm can be applied only in the case $t > t_0$.
2. Evaluate the statistics $Z(t)$ defined by (7).
3. Compute the value of $g(0, X_0, t, Z(t))$
4. Accept the hypothesis H_0 if $g(0, X_0, t, Z(t)) \le 1 - \alpha$, and the hypothesis H_1 otherwise.

Remark 2

Practically, the condition $t > t_0$ is not too restrictive, since for reasonable values of α values of t_0 are quite small.

Let us summarize the properties of the test in the following theorem.

Theorem 17

The test described in Algorithm 1 is unbiased and consistent as $t \to \infty$. *For the simple alternative* $\theta_1 > 0$ *and time* $t > t_0$ *the power of the test equals* $1 - g(\theta_1, X_0, t, c_t)$, *where* c_t *can be found from* (13).

Proof.

It follows from the monotonicity of g with respect to θ (see Lemma 15) that for any $\theta_1 > 0$

$$P(Z(t) \geq c_t) = 1 - g(\theta_1, X_0, t, c_t) > 1 - g(0, X_0, t, c_t) = \alpha.$$

This means that the test is unbiased. Evidently, for the simple alternative $\theta_1 > 0$ the power of the test equals $1 - g(\theta_1, X_0, t, c_t)$.
It follows from the convergence $c_t \to 0$, $t \to \infty$ (see Lemma 16) that $c_t < c$ for sufficiently large t and some constant $c \in (0, 1)$. Taking into account the formula (12) and Lemma 11 (i), we get as $t \to \infty$

$$\begin{aligned} 1 &\geq 1 - g(\theta_1, X_0, t, c_t) \geq 1 - g(\theta_1, X_0, t, c) \\ &= 2 - \Phi\left(\frac{e^{t^c} - X_0 e^{\theta_1 t}}{\sqrt{v(\theta_1, t)}}\right) - \Phi\left(\frac{e^{t^c} + X_0 e^{\theta_1 t}}{\sqrt{v(\theta_1, t)}}\right) \\ &\to 2 - \Phi\left(-\frac{X_0 \theta_1^H}{\sqrt{H\Gamma(2H)}}\right) - \Phi\left(\frac{X_0 \theta_1^H}{\sqrt{H\Gamma(2H)}}\right) = 1. \end{aligned}$$

Testing the hypothesis $H_0\colon \theta \geq \theta_0$ against $H_1\colon \theta \leq 0$

Let $\theta_0 \in (0, 1)$. Let us consider the problem of testing the hypothesis $H_0\colon \theta \geq \theta_0$ against alternative $H_1\colon \theta \leq 0$.

The next algorithm is based on the following results. They can be proved similarly to the previous subsection.

Lemma 18

Let $\alpha \in (0,1)$. There exists $\tilde{t}_0 > 1$ such that for all $t > \tilde{t}_0$ there exists a unique $\tilde{c}_t \in (0,1)$ such that

$$g(\theta_0, X_0, t, \tilde{c}_t) = \alpha. \tag{14}$$

In this case $\tilde{c}_t \to 1$ as $t \to \infty$.
The constant $\tilde{t}_0$ can be chosen as the largest $t > 1$ that satisfies at least one of the following two equalities

$$g(\theta_0, X_0, t, 0) = \alpha \quad \text{or} \quad g(\theta_0, X_0, t, 1) = \alpha.$$

Algorithm 2 The hypothesis $H_0\colon \theta \geq \theta_0$ against the alternative $H_1\colon \theta \leq 0$ can be tested as follows.

1. Find $\tilde{t}_0$ defined in Lemma 18. The algorithm can be applied only in the case $t > \tilde{t}_0$.
2. Evaluate the statistics $Z(t)$ defined by (7).
3. Compute the value of $g(\theta_0, X_0, t, Z(t))$, see (12).
4. Accept the hypothesis H_0 if $g(\theta_0, X_0, t, Z(t)) \geq \alpha$, and the hypothesis H_1 otherwise.

Theorem 19

The test described in Algorithm 2 is unbiased and consistent as $t \to \infty$. *For the simple alternative* $\theta_1 \le 0$ *and time* $t > \tilde{t}_0$ *the power of the test equals* $g(\theta_1, X_0, t, \tilde{c}_t)$, *where* $\tilde{c}_t$ *can be found from* (14).

Remark 3

It is possible to see from the numerics that if θ_0 is too close to zero, then for small H the condition $t > \tilde{t}_0$ does not hold for reasonable values of t.

Remark 4

If we have a confidence interval for θ, then the value of θ_0 can be chosen as a lower confidence bound (in the case when it is positive).

Continuous observations

Now we propose drift parameter estimators that work for any $H \in (0, 1)$. We consider continuous and discrete observations. Assume that a trajectory of $X = X(t)$ is observed over a finite time interval $t \in [0, T]$.

Lemma 20

Let $\theta < 0$. Then for any $p \geq 1$ there exist positive constants c_p and C_p such that

$$E\,|X_t|^p \leq c_p \quad \textit{for } t \geq 0, \tag{15}$$

$$E\,|X_t - X_s|^p \leq C_p\,|t-s|^{pH} \quad \textit{for } |t-s| \leq 1. \tag{16}$$

Proof.

By Lemmas 10 and 11 (ii), X_t is a Gaussian random variable,

$$|EX_t| = |X_0|\,e^{\theta t} \leq |x_0| < \infty,$$

$$\operatorname{var} X_t \to \frac{H\Gamma(2H)}{(-\theta)^{2H}} < \infty,$$

whence (15) follows.

Assume that $t \geq s \geq 0$ and $t - s \leq 1$. Let us show that

$$E\,|X_t - X_s| \leq C_1\,|t-s|^H\,, \quad \text{and} \quad E\,|X_t - X_s|^2 \leq C_2\,|t-s|^{2H}\,, \tag{17}$$

Proof.

By (6),

$$|X_t - X_s| \leq |\theta| \int_s^t |X_u| \, du + \left| B_t^H - B_s^H \right|.$$

Therefore, using (15), we get

$$\begin{aligned} E|X_t - X_s| &\leq |\theta| \int_s^t E|X_u| \, du + E\left| B_t^H - B_s^H \right| \\ &\leq c_1 |\theta| (t-s) + (t-s)^H \leq (c_1 |\theta| + 1)(t-s)^H, \end{aligned}$$

and

$$\begin{aligned} E(X_t - X_s)^2 &\leq 2|\theta|^2 E\left(\int_s^t |X_u| \, du \right)^2 + 2E\left(B_t^H - B_s^H \right)^2 \\ &\leq 2|\theta|^2 (t-s) \int_s^t E|X_u|^2 \, du + 2(t-s)^{2H} \\ &\leq 2|\theta|^2 c_2^2 (t-s)^2 + 2(t-s)^{2H} \leq 2\left(|\theta|^2 c_2^2 + 1 \right)(t-s)^{2H}. \end{aligned}$$

Proof.

Thus, (17) is proved. Since $X_t - X_s$ has a Gaussian distribution, (16) follows from (17) in the standard way. □

Lemma 21

For $\theta < 0$

$$\frac{1}{T}\int_0^T X_t^2\,dt \to \frac{H\Gamma(2H)}{(-\theta)^{2H}}.$$

as $T \to \infty$ *a. s. and in* L^2*.*

Proof.

It was proved in [4] that in this case the process $Y_t = \int_{-\infty}^t e^{\theta(t-s)}\,dB_s^H$ is Gaussian, stationary and ergodic. The integral with respect to the fractional Brownian motion here exists as a path-wise Riemann-Stieltjes integral, and can be calculated using integration by parts, see [4, Prop. A.1]. It follows from the ergodic theorem that

$$\frac{1}{T}\int_0^T Y_t^2\,dt \to EY_0^2$$

as $T \to \infty$ a. s. and in L^2. □

Proof.

The process X_t can be expressed as $X_t = Y_t - e^{\theta t}\eta_\theta$, where

$$\eta_\theta = \theta \int_{-\infty}^{0} e^{-\theta s} B_s^H \, ds - x_0$$

is a Gaussian random variable. □

Proof.

Using this representation, it is not hard to show that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T X_t^2 \, dt = \lim_{T\to\infty} \frac{1}{T} \int_0^T Y_t^2 \, dt = EY_0^2.$$

The value of the limit can be calculated applying Lemmas 10 and 11. □

Proof.

Indeed,

$$
\begin{aligned}
EY_0^2 &= E\left(\int_{-\infty}^0 e^{-\theta s}\, dB_s^H\right)^2 = \lim_{t\to-\infty} E\left(-e^{-\theta t} B_t^H + \theta \int_t^0 e^{-\theta s} B_s^H\, ds\right)^2 \\
&= \lim_{t\to\infty} E\left(-e^{\theta t} B_{-t}^H + \theta \int_0^t e^{\theta s} B_{-s}^H\, ds\right)^2 \\
&= \lim_{t\to\infty} e^{2\theta t}\left(E\left(B_{-t}^H\right)^2 - 2\theta e^{-\theta t}\int_0^t e^{\theta s} E B_{-t}^H B_{-s}^H\, ds\right. \\
&\quad \left. + \theta^2 e^{-2\theta t}\int_0^t \int_0^t e^{\theta(s+u)} E B_{-s}^H B_{-u}^H\, ds\, du\right) \\
&= \lim_{t\to\infty} e^{2\theta t} v(-\theta, t) = \frac{H\Gamma(2H)}{(-\theta)^{2H}}.
\end{aligned}
$$

□

Theorem 22

Let $H \in (0,1)$.

(i) *For* $\theta < 0$ *the estimator*

$$\hat{\theta}_T^{(1)} = -\left(\frac{1}{H\Gamma(2H)T}\int_0^T X_t^2 dt\right)^{-\frac{1}{2H}}$$

is strongly consistent as $T \to \infty$.

(ii) *For* $\theta > 0$ *the estimator*

$$\hat{\theta}_T^{(2)} = \frac{X_T^2}{2\int_0^T X_t^2 dt}$$

is strongly consistent as $T \to \infty$.

Proof.

(i) For $\theta < 0$ the result follows from Lemma 21.
(ii) If $\theta > 0$, then Lemma 12 implies the a. s. convergence

$$\frac{X_T^2}{e^{2\theta T}} \to \xi_\theta^2 \quad \text{as } T \to \infty. \tag{18}$$

Then, by L'Hôpital's rule,

$$\lim_{T\to\infty} \frac{\int_0^T X_t^2\,dt}{e^{2\theta T}} = \lim_{T\to\infty} \frac{X_T^2}{2\theta e^{2\theta T}} = \frac{\xi_\theta^2}{2\theta}. \tag{19}$$

Note that $0 < \xi_\theta^2 < \infty$ with probability 1, since ξ_θ is a Gaussian random variable. Combining (18) and (19), we get the convergence $\hat\theta_T^{(2)} \to \theta$ a. s. as $T \to \infty$. □

Remark 5

In the case $H \in [1/2, 1)$ the strong consistency of the estimators $\hat\theta_T^{(1)}$, $\hat\theta_T^{(2)}$ was proved in [7] and [1] respectively.

Discrete observations

Assume that a trajectory of $X = X(t)$ is observed at the points $t_{k,n} = \frac{k}{n}$, $0 \le k \le n^m$, $n \ge 1$ where $m > 1$ is some fixed number.

Theorem 23

Let $H \in (0,1)$, $m > 1$.

(*i*) *For $\theta < 0$ the estimator*

$$\hat{\theta}_n^{(3)}(m) = -\left(\frac{1}{H\Gamma(2H)n^m}\sum_{k=0}^{n^m-1} X_{k/n}^2\right)^{-\frac{1}{2H}}$$

is strongly consistent as $n \to \infty$.

(*ii*) *For $\theta > 0$ the estimator*

$$\hat{\theta}_n^{(4)}(m) = \frac{nX_{n^{m-1}}^2}{2\sum_{k=0}^{n^m-1} X_{k/n}^2}$$

is strongly consistent as $n \to \infty$.

Proof.

(i) Taking into account Theorem 22 (i), it suffices to prove the convergence

$$\zeta_n := \frac{1}{n^{m-1}} \int_0^{n^{m-1}} X_t^2 \, dt - \frac{1}{n^m} \sum_{k=0}^{n^m-1} X_{k/n}^2 \to 0 \quad \text{a.s. as } n \to \infty. \qquad (20)$$

Denote

$$Z_n(t) := \sum_{k=0}^{n^m-1} \left(X_t^2 - X_{k/n}^2 \right) 1_{[\frac{k}{n}, \frac{k+1}{n})}(t).$$

Then

$$\zeta_n = \frac{1}{n^{m-1}} \int_0^{n^{m-1}} Z_n(t) \, dt.$$

Using Lemma 20, one can show that

$$E \left| Z_n(t) \right|^p \leq K(p) n^{-pH}$$

for some constant $K(p) > 0$. Then by Hölder's inequality,

Proof.

$$E\,|\zeta_n|^p \le K(p) n^{-pH}.$$

Therefore by [10, Lemma 2.1], for all $\varepsilon > 0$ there exists a random variable η_ε such that

$$|\zeta_n| \le \eta_\varepsilon n^{-H+\varepsilon} \quad \text{a. s.}$$

for all $n \in \mathbb{N}$. Moreover, $E\,|\eta_\varepsilon|^p < \infty$ for all $p \ge 1$. This implies the convergence $\zeta_n \to 0$ a. s. as $n \to \infty$.
(ii) It follows from [11, Cor. 5.2(i)] that for $\theta > 0$

$$\frac{1}{n} \sum_{k=0}^{n^m - 1} X_{k/n}^2 = \int_0^{n^{m-1}} X_t^2\, dt + \vartheta_n,$$

where

$$\frac{\vartheta_n}{e^{2\theta n^{m-1}}} \to 0 \quad \text{a. s. as } n \to \infty.$$

□

Proof.

Combining this with Theorem 22 (*ii*) and (18), we get

$$\hat{\theta}_n^{(4)}(m) = \frac{X^2_{n^{m-1}}}{2\int_0^{n^{m-1}} X_t^2\,dt + 2\vartheta_n} = \left(\frac{1}{\hat{\theta}^{(2)}_{n^{m-1}}} + 2\cdot\frac{e^{2\theta n^{m-1}}}{X^2_{n^{m-1}}}\cdot\frac{\vartheta_n}{e^{2\theta n^{m-1}}}\right)^{-1} \to \theta$$

a. s. as $n \to \infty$. □

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