Elements of fractional calculus. Fractional Brownian motion. Wiener integration w.r.t. fBm. Some related fractional processes

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Outline

Introduction

Fractional calculus is one of the best tools to characterize long-memory processes and materials, anomalous diffusion, long-range interactions, long-term behaviors, power laws, allometric scaling laws, as well as the respective short-memory effects, especially in finance. So the corresponding mathematical models are, on the one hand, fractional processes, and on the other hand, fractional differential equations. Of course, this is a very simplified situation, because the number of fractional objects is much greater. There can be partial fractional differential equations etc. The evolutions of fractional processes behave in a much more complicated way so to study the corresponding dynamics is much more difficult.

But the study of such processes and equations is absolutely necessary, because fractality is inherent in almost all observed phenomena. It manifests itself in particular in the fact that, for example, the transmission of cellular signals cannot be described by differential equations with derivatives of an integer order, they are not smooth enough for this.

It can be said that the derivatives of the integer order that have been used to describe classical mechanical systems since Newton's, have become too much luxury at the present time, when we observe changes that occur too quickly in order to be residually smooth. So, derivatives and integrals with integer indices are irreplaceable in describing sufficiently smooth phenomena in finance, economics, modern technologies and natural phenomena (moreover, almost all of the above evolve according to very similar laws), and they are replaced by fractional order derivatives and fractional order integrals that describe some quasi-smoothness. Therefore, it is absolutely necessary to study and apply the elements of fractional calculus.

Remark 0.1

During the lectures, the technical details of the proofs and technical proofs will be omitted. We shall discuss only some principal details.

Fractional integrals

Let $\alpha > 0$ (in the most cases later $\alpha < 1$ but it is not obligatory). Denote the Riemann–Liouville left- and right- sided fractional integrals on (a, b) of order α as the operators I_{a+}^{α} and I_{b-}^{α} of the form

$$(I_{a+}^{\alpha} f)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1} dt,$$
$$(I_{b-}^{\alpha} f)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b f(t)(t-x)^{\alpha-1} dt.$$

We say that the function $f \in \mathcal{D}(I_{a+(b-)}^{\alpha})$ (symbol $\mathcal{D}(\cdot)$ denotes the domain of corresponding operator) if the corresponding integrals converge for almost all (a.a.) $x \in (a, b)$ (with respect to (w.r.t.) Lebesgue measure).

Fractional integrals

The Riemann–Liouville left- and right- sided fractional integrals on $\mathbb{R}$ are defined as

$$(I_+^\alpha f)(x) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} f(t)(x-t)^{\alpha-1} dt,$$

$$(I_-^\alpha f)(x) := \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} f(t)(t-x)^{\alpha-1} dt.$$

The function $f \in \mathcal{D}(I_\pm^\alpha)$ if the corresponding integrals converge for a.a. $x \in \mathbb{R}$. According to [SKM93], $L_p(\mathbb{R}) \subset \mathcal{D}(I_\pm^\alpha)$, $1 \le p < \frac{1}{\alpha}$.

Hardy–Littlewood theorem

Moreover, the following Hardy–Littlewood theorem holds.

Theorem 1.1 ([SKM93])

Let $1 \leq p, q < \infty$, $0 < \alpha < 1$. *Then the operators* $I_{\pm}^{\alpha}$ *are bounded from* $L_p(\mathbb{R})$ *to* $L_q(\mathbb{R})$ *if and only if* $1 < p < \frac{1}{\alpha}$ *and* $q = p(1 - \alpha p)^{-1}$. *It means, in particular, that for any* $1 < p < \frac{1}{\alpha}$ *and* $q = \frac{p}{1-\alpha p}$ *(or* $1/p - 1/q = \alpha$*) there exists a constant* $C_{p,q,\alpha}$ *such that*

$$\left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(u)||x - u|^{\alpha - 1} du \right)^q dx \right)^{\frac{1}{q}} \leq C_{p,q,\alpha} \|f\|_{L_p(\mathbb{R})}. \qquad (1)$$

Properties of fractional integration

Fractional integration admits the following composition formula:

$$I^{\alpha}_{a+} I^{\beta}_{a+} f = I^{\alpha+\beta}_{a+} f, \;\; I^{\alpha}_{b-} I^{\beta}_{b-} f = I^{\alpha+\beta}_{b-} f$$

for $f \in L_1[a,b]$. If $\alpha + \beta \geq 1$ we have these equalities at any point $x \in (a,b)$ otherwise they hold for a. a. x. Also,

$$I^{\alpha}_{\pm} I^{\beta}_{\pm} = I^{\alpha+\beta}_{\pm} f$$

for $f \in L_p(\mathbb{R})$, $\alpha, \beta > 0 \; \alpha + \beta < \frac{1}{p}$. For $f \in L_p[a,b]$, $g \in L_q[a,b]$, $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$, where $p > 1$, $q > 1$ for $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$ we have the following integration-by-parts formula

$$\int_a^b g(x)(I^{\alpha}_{a+} f)(x)dx = \int_a^b f(x)(I^{\alpha}_{b\_} g)(x)dx.$$

Let $f \in L_p(\mathbb{R})$, $g \in L_q(\mathbb{R})$, $p > 1, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$. Then

$$\int_{\mathbb{R}} g(x)(I^{\alpha}_{+} f)(x)dx = \int_{\mathbb{R}} f(x)(I^{\alpha}_{-} g)(x)dx. \tag{2}$$

Hölder continuous functions

Let $C^\lambda(T)$ be set of Hölder continuous functions $f : T \to \mathbb{R}$ of order λ, i.e.,

$$C^\lambda([a,b]) = \Big\{ f : [a,b] \to \mathbb{R} \,\Big|\, \|f\|_{[a,b],\lambda} := \sup_{t\in[a,b]} |f(t)| + \sup_{s,t\in[a,b]} |f(s)-f(t)|(t-s)^{-\lambda} < \infty \Big\}.$$

For $p \geq 1$ denote $I_\pm^\alpha(L_p(\mathbb{R}))$ the class of functions f that can be presented as the Riemann–Liouville integrals $f = I_\pm^\alpha \varphi$ for some $\varphi \in L_p(\mathbb{R})$, $p \geq 1$.

Lemma 1.2

If $\alpha > 0$, $\alpha p > 1$, then $I_\pm^\alpha(L_p(\mathbb{R})) \subset C^\lambda([a,b])$ for any $-\infty < a < b < \infty$ and $0 < \lambda \leq \alpha - \frac{1}{p}$.

The next result is evident.

Lemma 1.3

Let $0 < \alpha < 1$, $f \in L_p(\mathbb{R})$, $1 \leq p < \frac{1}{\alpha}$ *and* $I_{\pm}^{\alpha} f = 0$. *Then* $f(x) = 0$ *for a.a.* $x \in \mathbb{R}$.

Riemann–Liouville fractional derivatives

Lemma 1.3 supplies the uniqueness of such function φ that its fractional integral is some given function f, and for $0 < \alpha < 1$ this function φ coincides for a.a. $x \in \mathbb{R}$ with the left- (right-) sided Riemann–Liouville fractional derivative of f of order α:

$$(I_+^{-\alpha} f)(x) = (D_+^{\alpha} f)(x) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{-\infty}^{x} f(t)(x-t)^{-\alpha} dt,$$

$$(I_-^{-\alpha} f)(x) = (D_-^{\alpha} f)(x) := \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{x}^{\infty} f(t)(t-x)^{-\alpha} dt.$$

Riemann–Liouville derivatives on the interval

The Riemann–Liouville fractional derivatives can be considered on any interval $[a, b] \subset \mathbb{R}$ in the following way: we introduce the class $I^{\alpha}_{\pm}(L_p[a, b])$ of functions f that can be presented as $f = I^{\alpha}_{a+}\varphi$ ($f = I^{\alpha}_{b-}\varphi$) for $\varphi \in L_p[a, b]$, $p \geq 1$, and denote

$$(I^{-\alpha}_{a+}f)(x) = (D^{\alpha}_{a+}f)(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x f(t)(x-t)^{-\alpha}dt,$$

$$(I^{-\alpha}_{b-}f)(x) = (D^{\alpha}_{b-}f)(x) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^b f(t)(t-x)^{-\alpha}dt.$$

Weyl representation

In this case Riemann–Liouville fractional derivatives $D^{\alpha}_{a+}f$ and $D^{\alpha}_{b-}f$ admit the following Weyl representation (we suppose that $f = 0$ outside (a, b)):

$$
\begin{aligned}
(D^{\alpha}_{a+}f)(x) \quad &= \frac{1}{\Gamma(1-\alpha)}\Big(f(x)(x-a)^{-\alpha} \\
&+\alpha\int_a^x (f(x)-f(t))(x-t)^{-\alpha-1}dt\Big)\cdot \mathbf{1}_{(a,b)}(x), \\
(D^{\alpha}_{b-}f)(x) \quad &= \frac{1}{\Gamma(1-\alpha)}\Big(f(x)(b-x)^{-\alpha} \\
&+\alpha\int_x^b (f(x)-f(t))(t-x)^{-\alpha-1}dt\Big)\cdot \mathbf{1}_{(a,b)}(x),
\end{aligned}
$$

where the convergence of the integrals holds pointwise for a.a. $x \in (a, b)$ for $p = 1$ and in $L_p[a, b]$ for $p > 1$.

Properties of fractional derivatives

Composition formula for fractional derivatives has a form

$$D_{a+}^{\alpha} D_{a+}^{\beta} f = D_{a+}^{\alpha+\beta} f, \tag{3}$$

for $\alpha \geq 0$, $\beta \geq 0$, $f \in I_{a+}^{\alpha+\beta}(L_1(\mathbb{R}))$.

Integration-by-parts

Also, under the assumptions $0 < \alpha < 1$, $f \in I^{\alpha}_{a+}(L_p[a, b])$, $g \in I^{\alpha}_{b-}(L_q[a, b])$, $1/p + 1/q \leq 1 + \alpha$ we have following integration-by-parts formula

$$\int_a^b (D^{\alpha}_{a+} f)(x) g(x) dx = \int_a^b f(x)(D^{\alpha}_{b-} g)(x) dx. \tag{4}$$

Caputo derivatives

For $0 < \alpha < 1$ and $f \in C^1[a,b]$ the derivatives $D^\alpha_{a+} f$ and $D^\alpha_{b-} f$ exist, belong to $L_r[a,b]$ for $1 \le r < 1/\alpha$ and have a form

$$D^\alpha_{a+} f = \frac{1}{\Gamma(1-\alpha)} \left(f(a)(x-a)^{-\alpha} + \int_a^x f'(t)(x-t)^{-\alpha} dt \right),$$

$$D^\alpha_{b-} f = \frac{1}{\Gamma(1-\alpha)} \left(f(b)(b-x)^{-\alpha} - \int_x^b f'(t)(t-x)^{-\alpha} dt \right).$$

Let us consider only the integral in the latter formulas. We get Caputo derivatives:

$$D^{cap,\alpha}_{a+} \alpha f = \frac{1}{\Gamma(1-\alpha)} \int_a^x f'(t)(x-t)^{-\alpha} dt$$

$$= D^\alpha_{a+} f - \frac{1}{\Gamma(1-\alpha)} f(a)(x-a)^{-\alpha} +,$$

$$D^{cap,\alpha}_{b-} f = \frac{-1}{\Gamma(1-\alpha)} \int_x^b f'(t)(t-x)^{-\alpha} dt$$

$$= D^\alpha_{b-} f - \frac{1}{\Gamma(1-\alpha)} f(b)(b-x)^{-\alpha}.$$

Fractional integral of the indicator function

Let the general indicator function be given by

$$\mathbf{1}_{(a,b)}(t) = \begin{cases} 1, & a \le t < b, \\ -1, & b \le t < a, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 1.4

Let $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, $\alpha = H - \frac{1}{2}$. *Then for all* $t \in \mathbb{R}$

$$(I_-^\alpha \mathbf{1}_{(0,t)})(x) = \frac{1}{\Gamma(1+\alpha)}((t-x)_+^\alpha - (-x)_+^\alpha).$$

Proof.

Let $H \in (\frac{1}{2}, 1)$, and, for example, $x < 0 < t$ (other cases can be considered similarly). Then

$$\begin{aligned}(I_-^\alpha \mathbf{1}_{(0,t)})(x) &= \frac{1}{\Gamma(\alpha)} \int_x^\infty \mathbf{1}_{(0,t)}(u)(u-x)^{\alpha-1} du \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (u-x)^{\alpha-1} du = \frac{1}{\Gamma(\alpha+1)} \left((t-x)^\alpha - (-x)^\alpha\right). \quad (5)\end{aligned}$$

□

Remark 1.5

Obviously, $(I_+^\alpha \mathbf{1}_{(a,b)}(x)) = \frac{1}{\Gamma(1+\alpha)}((b-x)_+^\alpha - (a-x)_+^\alpha)$,
$-\infty < a < b < \infty$.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space.

Definition 2.1

(Two-sided, normalized) fractional Brownian motion (fBm) with Hurst index $H \in (0, 1)$ is a stochastic Gaussian process $B^H = \{B_t^H, t \in \mathbb{R}\}$ on $(\Omega, \mathcal{F}, P)$, having the properties

(i) $B_0^H = 0$;

(ii) $EB_t^H = 0, t \in \mathbb{R}$,

(iii) $EB_t^H B_s^H = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), s, t \in \mathbb{R}$.

Remark 2.2

Since $E(B_t^H - B_s^H)^2 = |t-s|^{2H}$ and the process is Gaussian, it has a continuous modification, according to Kolmogorov theorem, because for all $n \geq 1$ $E|B_t^H - B_s^H|^n = \frac{2^{\frac{n}{2}}}{\pi^{\frac{1}{2}}} \Gamma(\frac{n+1}{2})|t-s|^{nH}$. Moreover, the sufficient condition of Hölder continuity of order ϱ of the trajectories of X is $E|X_t - X_s|^n \leq C|t-s|^{1+n\varrho}$ for some $n > 0, \varrho > 0$. In our case we can put $\varrho = H - 1/n$, and to get Hölder property of the trajectories of fractional Brownian motion up to order H.

Remark 2.3

For $H = 1$ we set $B_t^H = B_t^1 = t\xi$, where ξ is standard normal random variable.

Remark 2.4

It is possible to consider fBm B_t^H only on $\mathbb{R}_+$ (one-sided fBm) with evident changes in Definition 2.1.

The characteristic function has a form

$$\varphi_\lambda(t) := E \exp\{i \sum_{k=1}^{n} \lambda_k B_{t_k}^H\} = \exp\left\{-\frac{1}{2}(C_t\lambda, \lambda)\right\},$$

where the matrix $C_t = (EB_{t_k}^H B_{t_i}^H)_{i,k=1}$, $(\cdot, \cdot)$ is inner product in $\mathbb{R}^n$. Therefore, from (iii) of Definition 2.1, for any $\alpha > 0$

$$\varphi_\lambda(\alpha t) = \exp\left\{-\frac{1}{2}\alpha^{2H}(C_t\lambda, \lambda)\right\}. \qquad (6)$$

Definition 2.5

Stochastic process $X = \{X_t, t \in \mathbb{R}\}$ is called β-self-similar if

$$\{X_{at}, t \in \mathbb{R}\} \stackrel{d}{=} \{a^\beta X_t, t \in \mathbb{R}\}$$

in the sense of finite-dimensional distributions.

It follows from Definition 2.5 and (6) that B^H is H-self-similar.
Note that

$$E(B_t^H - B_s^H)(B_u^H - B_v^H) = \frac{1}{2}(|s-u|^{2H} + |t-v|^{2H} - |t-u|^{2H} - |s-v|^{2H}). \quad (7)$$

It follows from (7) that the process B^H has stationary increments (evidently, it is not stationary itself). Let $H = \frac{1}{2}$. Then the increments of B^H are non-correlated consequently independent, so $B^H = W$ is a Wiener process.
In what follows, $\alpha = H - 1/2$.

For $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ and $t_1 < t_2 < t_3 < t_4$, from (7).

$$E(B^H_{t_4} - B^H_{t_3})(B^H_{t_2} - B^H_{t_1}) = H(2H-1)\int_{t_1}^{t_2}\int_{t_3}^{t_4}(u-v)^{2\alpha-1}dudv.$$

Therefore, the increments are positively correlated for $H \in (\frac{1}{2}, 1)$ and negatively correlated for $H \in (0, \frac{1}{2})$. Further, for any $n \in \mathbb{Z} \setminus \{0\}$ the autocovariance function

$$r(n) := EB^H_1(B^H_{n+1} - B^H_n) = H(2H-1)\int_0^1\int_n^{n+1}(u-v)^{2\alpha-1}\,du\,dv$$
$$\sim H(2H-1)|n|^{2\alpha-1}, \quad |n| \to \infty.$$

If $H \in (0, \frac{1}{2})$, then $\sum_{n\in\mathbb{Z}} |r(n)| \sim \sum_{n\in\mathbb{Z}\setminus\{0\}} |n|^{2\alpha-1} < \infty$.
If $H \in (\frac{1}{2}, 1)$, then $\sum_{n=1}^{\infty} |r(n)| \sim \sum_{n\in\mathbb{Z}\setminus\{0\}} |n|^{2\alpha-1} = \infty$. In this connection we say that for $H \in (\frac{1}{2}, 1)$ fBm B^H has the property of long-range dependence, while for $H \in (0, \frac{1}{2})$ fBm B^H has the property of short-range dependence.

Let $W=\{W_t, t\in\mathbb{R}\}$ be the two-sided Wiener process, i.e., Gaussian process with independent increments, $EW_t=0$, $EW_tW_s=s\wedge t$, $s,t\in\mathbb{R}$. Evidently, $W=B^{\frac{1}{2}}$. Denote

$$k_H(t,u):=(t-u)_+^{\alpha}-(-u)_+^{\alpha}=(I_-^{\alpha}\mathbf{1}_{(0,t)})(x),$$

$\alpha=H-\frac{1}{2}$. The following representation belongs to Mandelbrot and van Ness [MvN68].

Theorem 3.1

The process $\overline{B}^H=\{\overline{B}_t^H, t\in\mathbb{R}\}$ *with*

$$\overline{B}_t^H:=C_H^{(2)}\int_{\mathbb{R}}k_H(t,u)dW_u,\quad H\in\left(0,\frac{1}{2}\right)\cup\left(\frac{1}{2},1\right),$$

$$C_H^{(2)}=\left(\int_{\mathbb{R}_+}\left((1+s)^{\alpha}-s^{\alpha}\right)^2ds+\frac{1}{2H}\right)^{-\frac{1}{2}}=\frac{\left(2H\sin\pi H\Gamma(2H)\right)^{1/2}}{\Gamma(H+1/2)},$$

has a continuous modification that is a normalized two-sided fBm.

Define the operator

$$M_{\pm}^{H} f := \begin{cases} C_H^{(3)} I_{\pm}^{\alpha} f, & H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1), \\ f, & H = \frac{1}{2}, \end{cases} \quad (8)$$

where $C_H^{(3)} = C_H^{(2)} \Gamma(H + \frac{1}{2})$.
Consider the space $L_2^H(\mathbb{R}) := \{f : M_-^H f \in L_2(\mathbb{R})\}$ equipped with the norm $||f||_{L_2^H(\mathbb{R})} = ||M_-^H f||_{L_2(\mathbb{R})}$.

Definition 4.1

Let $f \in L_2^H(\mathbb{R})$. Then

$$I_H(f) := \int_{\mathbb{R}} f(s) dB_s^H := \int_{\mathbb{R}} (M_-^H f)(s) dW_s. \quad (9)$$

Here B_s^H and W_s are connected as in Theorem 3.1. As a particular case, consider stepwise function $f : \mathbb{R} \to \mathbb{R}$ that has a form

$$f(t) = \sum_{k=1}^{n} a_k \mathbf{1}_{[t_{k-1}, t_k)}(t),$$

Then, from the linearity of the operator M_-^H, the integral $I_H(f)$ equals

$$I_H(f) = \sum_{k=1}^{n} a_k \int_{\mathbb{R}} M_-^H \mathbf{1}_{[t_{k-1},t_k)}(s) dW_s = \sum_{k=1}^{n} a_k (B_{t_k}^H - B_{t_{k-1}}^H) \tag{10}$$

and it coincides with usual Riemann–Stieltjes sum. So, the question arises: in what sense can we consider formula (9) as the extension of the sum (10)? Note that for stepwise function

$$\begin{aligned} \|I_H(f)\|^2_{L_2(\Omega)} &= \sum_{i,k=1}^{n} a_i a_k \int_{\mathbb{R}} M_-^H \mathbf{1}_{[t_{k-1},t_k)}(x) M_-^H \mathbf{1}_{[t_{i-1},t_i)}(x) dx \\ &= \left\| M_-^H f \right\|^2_{L_2(\mathbb{R})} = H(2H-1) \int_{\mathbb{R}^2} f(u) f(v) \left| u - v \right|^{2\alpha - 1} du\, dv, \end{aligned} \tag{11}$$

where the last equality holds for $H \in (1/2, 1)$. One can see that the situation is very different for $H \in (0, 1/2)$ and $H \in (1/2, 1)$.

Nevertheless, there is a fact that holds for any $0 < H < 1$.

Lemma 4.2 ([Ben03a])

For any $0 < H < 1$ the linear span of the set $\{M_-^H \mathbf{1}_{(u,v)}, u, v \in \mathbb{R}\}$ is dense in $L_2(\mathbb{R})$.

Sometimes it is convenient to consider "one-sided" fBm $B^H = \{B_t^H, t \geq 0\}$ and to represent it as a functional of some Wiener process $B = \{B_t, t \geq 0\}$ of the form $B_t^H = \varphi(B_s, 0 \leq s \leq t)$. For this purpose, consider the following kernels

$$l_H(t,s) = C_H^{(5)} s^{-\alpha}(t-s)^{-\alpha} I_{\{0<s<t\}},$$
$$m_H(t,s) = C_H^{(6)} \Big[\big(\frac{t}{s}\big)^{\alpha}(t-s)^{\alpha} - \alpha s^{-\alpha} \int_s^t u^{\alpha-1}(u-s)^{\alpha} du \Big],$$

with $\alpha = H - \frac{1}{2}, H \in (0,1)$ and with the constants

$$C_H^{(5)} = \left(\frac{\Gamma(2-2\alpha)}{\Gamma(1-\alpha)^3 \Gamma(\alpha) 2H\alpha} \right)^{\frac{1}{2}}, \ C_H^{(6)} = \left(\frac{2H\Gamma(1-\alpha)}{\Gamma(1-2\alpha)\Gamma(\alpha+1)} \right)^{\frac{1}{2}}.$$

(i) Let $H \in (\frac{1}{2}, 1)$. Then for any $t > 0$

$$\int_0^t \int_0^t (t-u)^{-\alpha}(t-s)^{-\alpha} u^{-\alpha} s^{-\alpha} |u-s|^{2\alpha-1} du\, ds \sim t^{1-2\alpha} < \infty. \tag{12}$$

Therefore, we can consider the integral

$$I_t^H(l_H) = \int_0^t l_H(t,s)dB_s^H := \int_{\mathbb{R}} l_H(t,s)dB_s^H \\ = \int_{\mathbb{R}} (M_-^H l_H)(t,\cdot)(x)dW_x, \tag{13}$$

where W is the underlying Wiener process $\{W_x, x \in \mathbb{R}\}$. Similarly to (12), for any $0 < t < t'$ the scalar product equals

$$\mathbf{E} I_t^H(l_H) I_{t'}^H(l_H) = \big(l_H(t,\cdot), l_H(t',\cdot)\big)_{|R_H|,2} \\ = (C_H^{(5)})^2 2H\alpha \int_0^t (t-u)^{-\alpha} u^{-\alpha} \left(\int_0^{t'} (t'-s)^{-\alpha} s^{-\alpha} |u-s|^{2\alpha-1} ds \right) du \\ = (C_H^{(5)})^2 2H\alpha t^{1-2\alpha} B(\alpha, 1-\alpha) B(1-\alpha, 1-\alpha) = t^{1-2\alpha}. \tag{14}$$

From (13), $\left\{I_t^H, t \geq 0\right\}$ is a centered Gaussian process, and from (14), for any $0 < s < t \leq s' < t'$

$$\mathbf{E}\left(I_{t'}^H(l_H) - I_{s'}^H(l_H)\right)\left(I_t^H(l_H) - I_s^H(l_H)\right) \\ = t^{1-2\alpha} - t^{1-2\alpha} - s^{1-2\alpha} + s^{1-2\alpha} = 0,$$

i.e., the increments of $I_t^H(l_H)$ are non-correlated, hence, independent. Therefore, $I_t^H(l_H)$ is a martingale w.r.t. its natural filtration

$$\mathcal{F}_t^H := \sigma\left\{I_s^H(l_H), 0 \leq s \leq t\right\},$$

having independent increments and with quadratic characteristic $\left\langle I_t^H(l_H)\right\rangle = t^{1-2\alpha}, I_0^H(l_H) = 0.$

By Lévy theorem, there exists some Wiener process $B = \{B_t, t \geq 0\}$ such that

$$M_t^H := I_t^H(l_H) = (1 - 2\alpha)^{1/2} \int_0^t s^{-\alpha} dB_s. \tag{15}$$

The process M^H is called Molchan martingale, since it was considered originally in the papers [Mol69, MG69], see also [NVV99].
(ii) Now, let $H \in (0, \frac{1}{2})$.
Let the function f belong to $BV[0, T]$, the class of functions of bounded variation on $[0, T]$, and $f = 0$ outside $[0, T]$.Then the integral $\int_0^T f(s) dB_s^H$ exists for any $f \in BV[0, T]$ if we define it via integration by parts:

$$\int_0^T f(s) dB_s^H = f(t) B_t^H - \int_0^T B_s^H df(s).$$

Evidently, for any fixed $t > 0$ the kernel $l_H(t, \cdot) \in BV[0, t] \cap C[0, t]$, if $H \in (0, \frac{1}{2})$. Therefore,

$$
\begin{aligned}
I_t^H(l_H) &= \int_0^t l_H(t,s)dB_s^H = \int_0^t B_s^H dl_H(t,s) = \int_0^t B_s^H l_H'(t,s)ds \\
&= \alpha C_H^{(5)} \int_0^t B_s^H s^{-\alpha}(t-s)^{-\alpha-1}(t-2s)ds,
\end{aligned}
$$

and this integral is obviously a Gaussian random variable. We can easily calculate $EI_t^H(l_H)I_{t'}^H(l_H) = t^{1-2\alpha} = t^{2-2H}$ for any $0 < t < t'$, taking into the account the fact that l_H vanishes at the endpoints:

$$
\begin{aligned}
&\mathbf{E} I_t^H(l_H) I_{t'}^H(l_H) \\
&= \frac{1}{2}\int_0^t \int_0^{t'} (u^{2H} + s^{2H} - |u-s|^{2H}) l_H'(t,s) l_H'(t',u) du\, ds \\
&= -\frac{1}{2}\int_0^t \int_0^{t'} |u-s|^{2H} l_H'(t,s) l_H'(t',u) du\, ds
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{2}\int_0^t l'_H(t,s)\left(\int_0^s (s-u)^{2H} l'_H(t',u)du\right) ds \\
&\quad -\frac{1}{2}\int_0^t l'_H(t,s)\left(\int_s^{t'} (u-s)^{2H} l'_H(t',u)du\right) ds \\
&= -H\int_0^t l_H(t,s)\left(\int_0^s (s-u)^{2\alpha} l'_H(t',u)du\right) ds \\
&\quad + H\int_0^t l_H(t,s)\left(\int_s^{t'} (u-s)^{2\alpha} l'_H(t',u)du\right) ds \\
&= HC_H^{(5)}\int_0^t l_H(t,s) \\
&\quad \times \int_0^{t'} |u-s|^{2\alpha}\,\mathrm{sign}(u-s)u^{-\alpha-1}(t'-u)^{-\alpha-1}(t'-2u)du\,ds \qquad (16)
\end{aligned}
$$

According to [NVV99, formula (2.5)], the interior integral in the right-hand side of (16) equals for $s < t'$

$$\int_0^{t'} |u-s|^{2\alpha}\,\mathrm{sign}(u-s)u^{-\alpha-1}(t'-u)^{-\alpha-1}(t'-2u)du$$
$$= \left(H(C_H^{(5)})^2 B\,(1-\alpha, 1-\alpha)\right)^{-1},$$

whence $\mathbf{E} I_t^H(l_H) I_{t'}^H(l_H) = t^{2-2H}$. We can conclude, similarly to part (i), that $I_t^H(l_H)$ is a martingale according to its natural filtration, and

$$I_t^H(l_H) = (1-2\alpha)^{1/2} \int_0^t s^{-\alpha} dB_s$$

for some Wiener process B. Thus, we have proved the following result.

Theorem 5.1

Let B^H *be an fBm with* $H \in (0,1)$,

$$M_t^H = I_t^H(l_H) = \int_0^t l_H(t,s)dB_s^H. \qquad (17)$$

Then there exists Wiener process B *such that* $M_t^H = (1-2\alpha)^{1/2}\int_0^t s^{-\alpha}dB_s$. *It is clear that* $\sigma\left\{B_s^H, 0 \le s \le t\right\} = \sigma\left\{B_s, 0 \le s \le t\right\}$.

The converse relation can be obtained for any $H \in (0,1)$ by the similar way and has a form:

$$B_t^H = \int_0^t m(t,s)dW(s),$$

where

$$m_H(t,s) = C_H^{(6)}\Big[\big(\frac{t}{s}\big)^{\alpha}(t-s)^{\alpha} - \alpha s^{-\alpha}\int_s^t u^{\alpha-1}(u-s)^{\alpha}du\Big].$$

In the case $H > 1/2$ the kernel $m_H(t,s)$ can be simplified to $m_H(t,s) = \alpha C_H^{(6)} s^{-\alpha}\int_s^t u^{\alpha}(u-s)^{\alpha-1}du$.

Sub-fractional Brownian motion

Consider some related Gaussian processes.
Sub-fractional Brownian motion is a zero-mean Gaussian process $C^H = (C_t^H)_{t\geq 0}$ with parameter $H \in (0,1)$, such that its covariance function equals

$$EC_t^H C_s^H = t^{2H} + s^{2H} - \frac{1}{2}(|t+s|^{2H} + |t-s|^{2H}), \quad t,s \geq 0.$$

This process was introduced in [BGT04] in connection with the occupation time fluctuations of branching particle systems. In the case $H = 1/2$, it coincides with the standard Brownian motion: $C^{1/2} = B^{1/2}$. For $H \neq 1/2$, C^H is, in a sense, a process intermediate between the standard Brownian motion $B^{1/2}$ and the fBm B^H. Sub-fractional Brownian motion C^H has non-stationary increments, and its incremental covariance satisfies the inequalities (see [BGT04] for details)

$$|t-s|^{2H} \leq E|C_t^H - C_s^H|^2 \leq (2 - 2^{H-1})|t-s|^{2H}, \quad t,s \geq 0. \tag{18}$$

Bi-fractional Brownian motion

Let $B^{H,K} = (B_t^{H,K})_{t\geq 0}$, where $H \in (0,1)$, $K \in (0,1]$ are parameters, be a zero-mean Gaussian process with covariance function

$$EB_t^{H,K} B_s^{H,K} = 2^{-K}\big((t^{2H} + s^{2H})^K - |t-s|^{2HK}\big), \quad t,s \geq 0.$$

This process can be considered as an extension of the fBm, the latter being a special case when $K = 1$ (see [HV03, RT006, LN09]). The process $B^{H,K}$ satisfies the following version of (18):

$$2^{-K}|t-s|^{2HK} \leq E|B_t^{H,K} - B_s^{H,K}|^2 \leq 2^{1-K}|t-s|^{2HK}, \quad t,s \geq 0.$$

Mixed fractional Brownian motion

Consider a zero-mean Gaussian process of the form

$$M_t^H = W_t + B_t^H, t \geq 0,$$

where W and B^H are independent stochastic processes, W is a standard Brownian motion, and B^H is a fractional Brownian motion. Such process is called a mixed fractional Brownian motion. It was considered in detail by Cheridito in [Cher01].

References I

- Bender, C.: Integration with respect to a fractional Brownian motion and related market models. PhD Thesis, Hartung-Gorre Verlag, Konstanz (2003)
- Bisaglia, L., Guegan, D.: A review of techniques of estimation in long memory processes: application to intraday data. Comp. Stat. Data Analysis, **26**, 61–81 (1997)
- T. Bojdecki, L.G. Gorostiza, and A. Talarczyk, Sub-fractional Brownian motion and its relation to occupation times, Statist. Probab. Lett. 69 (2004), pp. 405–419.
- Cheridito, P. (2001). Mixed fractional Brownian motion. Bernoulli, 7(6), 913-934.
- Ciesielski, Z., Kerkyacharian, G., Roynette, B.: Quelques espaces fouctionnels associes á des processus gaussiens. Studia Mat., **107**, 171–204 (1993)

References II

- Dzhaparidze, K., van Zanten, H.: Krein's spectral theory and the Paley-Wiener expansion for fractional Brownian motion. Ann. Prob., **33**, 620–644 (2005)
- Gradshteyn, I. S.; Ryzhik, I. M.: Table of Integrals, Series, and Products. Academic Press, New York (1980)
- Grippenberg, G., Norros, I.: On the prediction of fractional Brownian motion. J. Appl. Prob., **33**, 400–410 (1996)
- C. Houdré and J. Villa, An example of infinite dimensional quasi-helix, Contemp. Math. 336 (2003), pp. 195–202.
- Kleptsyna, M.L., Le Breton, A., Roubaud, M.C.: General approach to filtering with fractional Brownian noises—application to linear systems. Stoch. Stoch. Rep., **71**, 119-140 (2000)

References III

- Kondurar, V.: Sur l'intégrale de Stieltjes. Recueil Math., **2**, 361–366 (1937)
- Le Breton, A.: Filtering and parameter estimation in a simple linear system driven by a fractional Brownian motion. Stat. Prob. Lett., **38**, 263–274 (1998)
- P. Lei and D. Nualart, *A decomposition of the bifractional Brownian motion and some applications,* Statist. Probab. Lett. 79 (2009), pp. 619–624.
- Liptser, R. S., Shiryaev, A. N.: Statistics of random processes. 1: General theory. Applications of Mathematics. 5. Berlin: Springer (2001)
- Mandelbrot, B.B., van Ness J.W.: Fractional Brownian motions, fractional noices and applications. SIAM Review, **10**, 422–437 (1968)

References IV

- Mishura,Y. Stochastic calculus for fractional Brownian motion and related processes (Vol. 1929). Springer Science & Business Media (2008)
- Molchan, G.: Gaussian processes with spectra which are asymptotically equivalent to a power of λ. Theory Prob. Appl. **14**, 530–532 (1969)
- Molchan, G., Golosov, J.: Gaussian stationary processes with asymptotic power spectrum. Soviet Math. Dokl. **10**, 134–137 (1969)
- Norros, I., Valkeila, E., Virtamo, J.: An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions. Bernoulli **5(4)**, 571–587 (1999)
- Nualart, D. Răşcanu, A.: Differential equations driven by fractional Brownian motion, Collect. Math., **53**, 55–81 (2000)

References V

- Pipiras, V., Taqqu, M.: Integration questions related to fractional Brownian motion. Prob. Theory Rel. Fields, **118**, 251–291 (2000)
- F. Russo and C.A. Tudor, On bifractional Brownian motion, Stochastic Process. Appl. 116 (2006), pp. 830–856.
- Samko, S. G., Kilbas, A. A., Marichev, O.I.: Fractional Integrals and Derivatives. Theory and Applications. Gordon and Breach Science Publishers, New York (1993)
- Samorodnitsky, G., Taqqu, M.S.: Stable Non-Gaussian Random Processes. Champan & Hall, New York (1994)
- Zähle, M.: Integration with respect to fractal functions and stochastic calculus. I. Prob. Theory Rel. Fields, **111**, 333–374 (1998)

References VI

- Zähle, M.: On the link between fractional and stochastic calculcus. In: Grauel, H., Gundlach, M. (eds.) Stochastic Dynamics, Springer, 305–325 (1999)
- Zähle, M.: Integraton with respect to fractal functions and stochastic calculus. II. Math. Nachr., **225**, 145–183 (2001)