On words that are concise in residually finite groups

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It remains unknown whether the word w^q is actually concise in the class of all groups.

We say that a word w is boundedly concise in a class of groups X if for every integer m there exists a number $\nu = \nu(X, w, m)$ such that whenever $|G_w| \leq m$ for a group $G \in X$ it always follows that $|w(G)| \leq \nu$.

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Recall that the word γ_k is defined inductively by the formulae

$$\gamma_1 = x_1, \qquad \gamma_k = [\gamma_{k-1}, x_k] = [x_1, \ldots, x_k], \quad \text{for } k \geq 2.$$

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Let w be a group-word and G a group. A subgroup $N \leq G$ will be called w-subgroup if N is generated by cyclic subgroups contained in G_w. By weight of a multilinear commutator we mean the number of variables involved in the word. It is clear that any multilinear commutator w of weight $k \geq 2$ can be written in the form $w = [w_1, w_2]$ where w_1 and w_2 are multilinear commutators of smaller weights.

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Let G be a perfect group and A a normal abelian subgroup of G. Then $[A, G]$ is a w-subgroup for any multilinear commutator w.

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Lemma

Let $w = [w_1, w_2]$ be a multilinear commutator word. Let K be a normal subgroup of a group G and suppose that K is nilpotent of class two. If $K/Z(K)$ is a w_1-subgroup in $G/Z(K)$ and if it is generated by w_2-values in $G/Z(K)$, then K' is a w-subgroup in G.

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PROPOSITION: Let w be a multilinear commutator. There exist a (d, n, w)-bounded integer s and a (d, n)-bounded integer h with the following property: Let G be a group having a normal soluble subgroup of finite index n and derived length d. Then G has a series of normal subgroups

$$1 = T_1 \leq T_2 \leq \cdots \leq T_s = w(G)$$

such that every quotient T_i/T_{i-1} is an abelian w-subgroup in G/T_{i-1}, except possibly one quotient whose order is at most h.

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The proof is somewhat technical. We will not discuss the details.

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It is interesting to compare the proposition that if G is soluble-by-finite then G has a series $1 = T_1 \leq T_2 \leq \cdots \leq T_s = w(G)$ such that every quotient T_i/T_{i-1} is an abelian w-subgroup in G/T_{i-1}, except possibly one quotient whose order is finite with other results of similar nature.

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If G is soluble and $w = \delta_k$ is the derived word, then G has a series $1 = T_1 \leq T_2 \leq \cdots \leq T_s = G^{(k)}$ such that every quotient T_i/T_{i-1} is an abelian w-subgroup and s depends only on k (S. Brazil, A. Krasilnikov, P. Shumyatsky, 2006).

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If G is soluble and w is any multilinear commutator, then G has a series $1 = T_1 \leq T_2 \leq \cdots \leq T_s = w(G)$ such that every quotient T_i/T_{i-1} is an abelian w-subgroup and s depends only on w (Fernández-Alcober and Morigi, 2010) .

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The proofs of our main results are based on the techniques that Zelmanov created in his solution of the restricted Burnside problem. Recall that the restricted Burnside problem was whether or not the order of a finite m-generated group G of exponent e is bounded in terms of m and e only. In 1957 Hall and Higman reduced the problem to the case where G is a p-group for some prime p. Their reduction theorem used the (future at that time) classification of finite simple groups and the representation theory. The case where G is a p-group remained open for more than 30 years. Then Zelmanov solved the problem in 1989. The solution of the RBP for p-groups was based on Lie methods created in the late 30s by Magnus and Zassenhaus.

One immediate corollary of the solution of the RBP is the fact that any residually finite group of finite exponent is locally finite.

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Theorem

Let q be a prime-power and w a multilinear commutator word. Assume that G is a residually finite group such that any w-value in G has order dividing q. Then the verbal subgroup $w(G)$ is locally finite.

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Let G be a residually finite group in which the word $v = w^q$ has only finitely many values. It is sufficient to show that $v(G)$ is periodic. Choose a normal subgroup K in G such that the index $[G : K]$ is finite and $v(K) = 1$ (such a subgroup exists because G is residually finite).

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PROPOSITION: *Let $m, k \geq 1$ and q be a p-power for some prime p. Assume that G is a finite m-generator group such that $[x_1, \ldots, x_k]^q = 1$ for all $x_1, \ldots, x_k \in G$. Then the exponent of $\gamma_k(G)$ is (k, m, q)-bounded.*

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This is a result in the spirit of the RBP and the proof is based on Lie-theoretic results of Zelmanov.

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Assume that G is a finite group with at most m values of the word $v = w^q$. Without loss of generality we can assume that G is generated by at most mk elements. Indeed, let $\{u_1, \ldots, u_m\}$ be the set of all v-values in G. Write $u_i = [g_{i1}, \ldots, g_{ik}]^q$ for $i = 1, \ldots, m$. Let K be the subgroup of G generated by $g_{11}, g_{12}, \ldots, g_{mk}$ and note that $v(G) = v(K)$. Hence, without loss of generality we can assume that G is generated by $g_{11}, g_{12}, \ldots, g_{mk}$.

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